Fleeting Route Design with Uncertainty

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ABSTRACT

During COVID-19, the air industry has shrunk due to drastically reduced demand and flight bans. Air transport network optimization is significant in fleet routing design, which helps the air company to make a decision to open the new routes while maximizing the profit. We want to study when an area is subjected to stability shocks that prevent air trac from entering, how air planners comprehensively consider the factors to design new fleeting routes while optimizing the profit. In this research, we start by developing a baseline model and then a networked model, both theoretical, to simulate the situation for an air company and provide optimization results. We find a simple situation to determine: if a route has a positive maximized expected profit, we will decide to open the route. The optimal results obtained from the model are proved to be optimal by mathematics analysis in the study. Several potential future study directions relevant to the research are discussed, including stochastic demand and network connection, which could be better explain the situation of air transport network design and thus be more applicable in reality.

Introduction

Fleet routing is a significant part of the formation of the air transport network. Several factors must be considered in designing new air routes such as hub location, passenger demand, policy, and resource availability to maximize profit. Air planners try to find an optimal situation for the network using these considerations. Other factors that may influence route design are resources, price, and fixed cost. However, some factors can also affect the network. Realistically, unforeseen circumstances happen in the air transport industry, such as changes in passenger demand and promulgated policies during pandemics (Barla & Constantatos, 2000).

The transport industry was severely impacted by the COVID-19 pandemic, particularly air transport. Due to the drastically reduced demand for passengers and country-wide flight bans, airline companies were forced to cut routes, almost grounding entire fleets (Iacus et al., 2020). Major carriers even saw as much as a 60% decline in capacity (Josephs, 2020). As a result, the air company planners had to design fleet routes with careful consideration to maximize their expected profit. This research explores the factors that influence the decision of the planner on designing a fleet route in several settings.

To develop a realistically applicable model, the study will identify and analyze several uncertainties within the industry. The primary aim of this research is to study how to form an air transport network in a practical situation with additional uncertainty to be considered.

Like the model developed by Yang (2010), this study's model focuses on network design. But there are some differences in the problems explored by the research. The study attempts to solve the demand of passengers on both sides of routes and the network structure with constraints (Yang, 2010). The study assumes that there is a point-to-point network structure with two hubs on one route. An air transport planner can observe the price, cost, and probability of success for all air routes. The research will analyze the simplest scenario to understand the model. In this research, the planner can choose the air route between two locations i and j and naturally do so in a way that maximizes profit. The study plans to focus on the price of the ticket, the cost of designing and operating the air routes, and the probability of the successful flight of the routes. Mathematical analysis will include relevant factors (the expectation, utility, and profit function) to obtain the baseline model. From this base model, the research will extend to involve more constraints and multiple hubs to develop a model with networks that better apply to the real world. Furthermore, the study wants to verify the results from the optimization under a simple constraint on the planner.

The study begins with a literature review to explore research about network design with similar problems. The next section establishes the methods and includes notations, functions, and assumptions that will be applied in the models. The results are presented in three subsections that describe the approaches used to obtain the baseline model, the development of the baseline model with additional constraints to the network model, and the optimization results. Finally, the final portion of the study discusses the conclusions from the modeling and provides some potential directions for this research.

Literature Review

This section summarizes the findings from the literature, which offer related information and specific details relevant to air transport network optimization. The focus is on background, theorem, mathematical method, and model construction. The research examines papers that study demand shocks, such as those related to disaster or pandemic-induced shocks. Additionally, this study seeks to identify previous research about air transport, which could inform the elements this study needs to consider in its model and to have a more detailed view of these topics. Moreover, the study looks through the research with network optimization to identify the ideal math method to handle the uncertainty and the way to construct the networked model. The research found several sources that could help to better understand the background of the topic and have a detailed view of the application of the concept. The review discusses the following order of topics: (1) air transport models, (2) network model construction, and (3) methods for solving optimization problems.

Air Transport Models

This section discusses the research on air transport models, which could offer guidelines to understand the air route design and the elements that need to be considered.

Relevant models and basic knowledge on air transport are presented by Yang (2010) and Soylu and Katip (2019), which helps to analyze the topics and develop the model. The topic of these two research is airline network design, and the model in this study is about network optimization—both topics are closely related to the search. The proposed research could apply the methods and parameters from other authors to build the new models. Additionally, both studies identify the airline network design problems, giving us a clear view of how to modify and improve the research questions. Our research will apply the guidance to improve and analyze the topics in detail. In Yang (2010), the author applies the expectation to handle the uncertainty in the objective function. This research will use the expectation function to process the uncertainty in the model in the following model sections.

Network Model Construction

This section explores network model construction and the methods used to develop the model.

Yan and Tseng (2002) offer some relevant model constructions by describing their model; they provide guidance on how to develop a networked model from step-by-step analysis. This research will apply Yan and Tseng's method in the model development since the research plan is to analyze the topics and develop the model from the baseline then add complexities, which will discuss more in the following model sections. Yan and Tseng's way of building the model could potentially be applied to the proposed research. Yan & Tseng's study gives an insight view about the elements that affected air transport, such as nodes and demand (2002).

Additionally, the research by Barbarosoğlu and Arda (2004) observes some background information and related models. Their article explores the model of material flow over an urban transportation network during a disaster, and they aimed to develop an efficient response to the disaster via a decision-making process (Barbarosoğlu & Arda, 2004). The timing and impact of a disaster are hard to predict; this is a primary concern of trying to prepare for world pandemics. The article considers the uncertainty and the vulnerability of the transportation system, which influences the supply, capacity, and demand. The uncertainty will be included in the proposed research.

Methods for Solving Optimization Problems

This part will discuss the research of optimization. The proposed research wants to find a method that could be applied when trying to obtain the result of the optimization.

Liu, et al. (2018) give some relevant optimization methods and models. The authors developed a model to assign the new air fleeting under stochastic demand to maximize the expected total profit (Liu et al., 2018). This article assumes the situation that the decision-maker is risk-averse (Liu et al., 2018). The proposed research will likely have some discussion about the type of decision-maker that could be based upon this article. The optimization model developed by the authors applies the decision variable in the mixed-integer programming model (Liu, et al., 2018), which could provide some guidance as we develop our model. The model developed by Liu (2018) makes the decision variable in the requirement 0 and 1.

The research discussed in the literature review section generalizes a detailed view of the air transport network and the elements in it. This research includes the nodes, routes, demand and supplies, and how to build relationships among them. And these studies inform that the expectation function could help us to handle the uncertainty. Moreover, this research has guidance about how to construct a networked model. Additionally, the research offered the method of how to design a model from baseline to more complex through step-by-step analysis.

Methods

Model

The section will explore and present the notations, functions, and assumptions necessary to build a mathematical understanding of air transport. To begin, the notations used for the research models are listed in Table 1.

The airline planner has to decide whether or not to open a new route between any pairs of location i and location j in a network with n hubs. θ_{ij} satisfies $0 \leq \theta_{ij} \leq 1$. γ denotes the constant relative risk aversion ($\gamma \neq 1$) in the utility function. $\gamma = 0$ means the decision-maker is risk-neutral. $\gamma > 0$ means the decision-maker is risk-averse. For the demand function $N_{ij} = a - b * P_{ij}, b > 0$. The decision variable for air-route design is denoted by $X \in [0,1]^{n \times n}$. If and only if $X_{ij} = 1$, the air planner decides to set up the flight from i to j. Of course, if $X_{ij} = 0$, the planner decides not to build the new routes.

Two cases in the model will be discussed: "not open the route" and "open the route". Under Jensen's inequality ($E(U(X)) \leq U(E(X))$), the baseline model will include the profit, utility, and expectation functions. First, the study will apply the profit function ($\pi(X_{ij})$), which calculates the profit for each route by using the total revenue of selling the tickets minus the total cost of setting the routes. Then, the utility function $U(Y) = \frac{Y^{1-\gamma}}{1-\gamma}$ will be used to calculate the utility of the profit for each route. Finally, the expected function will be applied to find the expected utility of a route.

Table 1. Two List of Variables in Air Route Design

n	the number of hubs in the network
i	any departure hub in the network
j	any destination hub in the network
θij	the probability of a successful flight from i to j
Nij	the total demand for the flight from i to j
Cij	the fixed cost of setting up and operating the air route from i to j
Pij	the price charged for the ticket of the air route from i to j
γ	the constant relative risk aversion in the utility function
Y	the profit in utility function
a	the outside factors other than price that affect demand
b	the elasticity of demand
$X \in [0,1]^{n\times n}$	the decision variable for air-route design

Case 1: Not open the route

The study first considers the case where the planner decides to not open the route. In this case, $X_{ij} = 0$; therefore, no profit can be made. When the planner decides not to open the new route from A to B: $X_{ij} = 0$. The profit function is shown in Equation (1), the utility function is shown in Equation (2), and the expected value of utility function is shown in Equation (3).

$$\pi(X_{ij}) = 0 \quad (1)$$

$$U(Y) = U(\pi(X_{ij})) = \frac{Y^{1-\gamma}}{1-\gamma} = \frac{\pi(X_{ij})^{1-\gamma}}{1-\gamma} = 0 \quad (2)$$

$$E(U(Y)) = E(U(\pi(X_{ij}))) = 0 \quad (3)$$

Case 2: Open the route

When the planner decides to set up the new route from i to j, $X_{ij} = 1$. The profit and utility can be calculated in the Equation (4) and Equation (5) respectively.

$$\pi(X_{ij}) = \begin{cases} N_{ij} * P_{ij} - C_{ij}, & probability \ \ \theta_{ij} \\ -C_{ij}, & probability \ \ 1-\theta_{ij} \end{cases} \quad (4)$$

$$U(Y) = U(\pi(X_{ij})) = \frac{Y^{1-\gamma}}{1-\gamma} = \frac{\pi(X_{ij})^{1-\gamma}}{1-\gamma} = \begin{cases} \frac{(N_{ij}*P_{ij}-C_{ij})^{1-\gamma}}{1-\gamma}, & probability \ \ \theta_{ij} \\ \frac{(-C_{ij})^{1-\gamma}}{1-\gamma}, & probability \ \ 1-\theta_{ij} \end{cases} \quad (5)$$

Therefore, the expected utility of profit can be computed by equation 6:

$$E(U(\pi(X_{ij}))) = \theta_{ij} * \frac{(N_{ij}*P_{ij}-C_{ij})^{1-\gamma}}{1-\gamma} + (1-\theta_{ij}) * \frac{(-C_{ij})^{1-\gamma}}{1-\gamma} \quad (6)$$

The study will include two assumptions in the model: Risk neutral air planner and Linear demand function.

Assumption 1: Risk-neutral air planner

Assume that the air planner is risk neutral ($\gamma = 0$). When $\gamma = 0$, it could conclude that the utility function is equal to the profit function. Moreover, by this assumption, the study expects the sum of the utility of all routes is the same as the sum of the expected utility of each route ($E(U(\Sigma Y)) = \Sigma E(U(Y))$). Under this assumption, the results of the functions obtained above in Equation (6) will change, input $\gamma = 0$ to the Case 1 and Case 2.

Case 1 ($X_{ij} = 0$)
The utility function is equal to the profit function, as $\gamma = 0$. The profit function is shown as Equation (7), and the expected value of the utility of profit is computed in Equation (8):

$$U(Y) = U(\pi(X_{ij})) = 0 \quad (7)$$

$$E(U(Y)) = E(U(\pi(X_{ij}))) = E(\pi(X_{ij})) = 0 \quad (8)$$

Case 2 ($X_{ij} = 1$)
Considering the utility function is equal to the profit function. Input the $\gamma = 0$ to Equation (5), the utility result is shown in the Equation (9):

$$U(Y) = U(\pi(X_{ij})) = \pi(X_{ij}) = \begin{cases} N_{ij} * P_{ij} - C_{ij}, & probability \ \ \theta_{ij} \\ -C_{ij}, & probability \ \ 1-\theta_{ij} \end{cases} \quad (9)$$

If we apply equation (6) and the assumption $\gamma = 0$, equation (10) becomes:

$$E(U(\pi(X_{ij}))) = \theta_{ij} * N_{ij} * P_{ij} - C_{ij} \quad (10)$$

Assumption 2: Linear demand function

The air planner could decide on the price level P_{ij}, which could further influence the demand N_{ij} in Equation (11). The research assumes that there is a linear relationship between the demand N_{ij} and ticket price P_{ij}.

$$N_{ij} = a - b * P_{ij} \quad (11)$$

In this case, *a* and *b* are constants, and we assume that there is a negative linear relationship between N_{ij} (demand) and

P_{ij}(price level), so let $b > 0$. Under this assumption, the function is relevant to the P_{ij}(price level).

For Case 1, $X_{ij} = 0$, from equation (8), thereby equation 12 is equal to 0 as well:

$$E(U(\pi(X_{ij}))) = 0 \tag{12}$$

For Case 2, $X_{ij} = 1$, from equation (10) and linear demand function (11), the expected utility of profit is computed in Equation (13):

$$E(U(\pi(X_{ij}))) = \theta_{ij} * (a - b * P_{ij}) * P_{ij} - C_{ij} \tag{13}$$

Results

The study applies optimization analysis to develop the baseline model and then adds several constraints to build a networked model. Finally, mathematical proofs are used to identify the optimal solution for a planner choosing to open a subset of possible routes in a network.

Baseline Model

This section will involve the simplest case ($n = 2$) for airline routes design as a baseline model. The air planner must decide whether to set up a new air route between two locations i and j by the standard technique of maximizing expected profit. Based on the functions obtained above, the research aimed to construct the model, explore the optimization result, and make the best decision.

The research explored expected profit in two possible cases in the model. Then two assumptions are applied to improve the functions. The model is then constructed, and the three possibilities (indifference, open, and not open) are explored. Finally, the optimization results are discussed.

To further explore the model and decide based on the optimization in this part, the profit of the functions for "Not open the route"($X_{ij} = 0$) and "Open the route"($X_{ij} = 1$) were explored.

Compare: Case 1 ($X_{ij} = 0$), $E(U(\pi(X_{ij}))) = 0$ and Case 2 ($X_{ij} = 1$), $E(U(\pi(X_{ij}))) = \theta_{ij} * (a - b * P_{ij}) * P_{ij} - C_{ij}$

In other words, it must compare 0 with $\theta_{ij} * (a - b * P_{ij}) * P_{ij} - C_{ij}$. Multiple values of P_i can be used to find the profit. Then these profits can be compared to 0 to find whether an air planner needs to decide to open the route.

Assume: The planner tries to maximize the profit. So the value of the parameters can be adjusted to find a maximized profit. In the model, the air planner could adjust the P_{ij}(price level) to influence N_{ij}(demand) and finally influence the profit. And assume that air planners will try to pursue the highest-profit route. From equation (13), the optimization model can be adjusted by sum to form equation (14):

$$\max_{X} \sum_{i,j=1}^{n} E(U(\pi(X_{ij} = 1))) = \theta_{ij} * (a - b * P_{ij}) * P_{ij} - C_{ij} \tag{14}$$

To find the cases that could have the maximized value of this equation (14), a quadratic equation about the variable of P_i (price level) built. Simplify the equation (14) and rearrange the parameters, the result is shown as the equation (15):

$$\max_X \sum_{i,j=1}^{n} E(U(\pi(X_{ij}=1))) = b*\theta_{ij}*P_{ij}^2 - a*\theta_{ij}*P_{ij} - C_{ij} \tag{15}$$

When $P_{ij} = \frac{a}{2b}$, the profit defined by equation (14) is maximized and expressed in Equation(16).

$$\pi_{max} = -C_{ij} + \frac{a^2\theta_{ij}}{4b} \tag{16}$$

Compare the highest profit it obtains to $E(U(\pi(X_{ij}))) = 0$, $X_{ij} = 0$ to decide whether the air planner could decide to open this route. Let the maximized profit obtained from the quadratic equation above be denoted π_{max}. The route with the highest profit will be the best choice among all possible ticket prices. Choose the situation that has been known the estimated highest profit made by the optimal price. Then the π_{max} is compared to 0. The expected profit of not opening the route can be evaluated in the following outcomes.

Decision 1: indifferent

When $\pi_{max} = 0$, equation 16 becomes equation 17.

$$C_{ij} = \frac{a^2\theta_{ij}}{4b} \tag{17}$$

Hence if $C_{ij} = \frac{a^2\theta_{ij}}{4b}$, the air planner will make an indifferent decision between opening or not opening the new route. Otherwise, the indifferent situation does not exist.

Decision 2: open the routes

When $\pi_{max} > 0$, the equation (16) for maximized profit becomes equation (18), and the planner decides to open the route.

$$C_{ij} < \frac{a^2\theta_{ij}}{4b} \tag{18}$$

In this case, the air company could obtain the highest profit of all possible routes. Otherwise, the “open route" case does not exist.

Decision 3: not open the routes

When $\pi_{max} < 0$, the equation (16) for maximized profit becomes equation (19), and the planner decides not to open the route.

$$C_{ij} < \frac{a^2\theta_{ij}}{4b} \tag{19}$$

Since the air company will tend to make a loss on the new route even considering the most-profited route, the planner is likely to suggest not opening the route. Otherwise, the "not open the routes" situation does not exist.

Networked Model

The baseline model above focuses on the simplest case that designs a route between $n = 2$ locations and then makes the decision whether to open a route based on the comparison of expected profit. To explore what will happen if there are multiple locations in the network, we must assume that there are multiple pairs of departure and destination hubs in a network.

The research will try to consider some natural constraints as follows. It wants to consider the maximum number of flights (air route) allowed to be put into use and the passenger capacity in the model. Let K represent the maximum number of flights allowed by the planner. Denote N as the passenger capacity for the whole network.

The research will start by analyzing the results obtained from the baseline model of the single pair, then apply the relevant results to explore the multiple pairs in the networks. Since the decision-maker is risk-neutral, it could directly apply the results developed in the baseline model.

For multiple pairs case: Based on the model from the simplest case, the research develops the model from a single pair (i, j) to every possible pair $(i, j) \in [1, \ldots, n]^2$ in the network. Let the assumptions made for the baseline model still work for this model with networks: risk neutral ($\gamma = 0$), and demand have a linear negative relationship with price level ($N_{ij} = a - b * P_{ij}$). Assuming that in the network, the demand and price level of the different routes hold the same relationship (a and b work for all the routes in the network).

From the profit equation of Case 1: $X_{ij} = 0$ (equation (1)). And from the profit equation of Case 2: $X_{ij} = 1$ (equation (4)). The Equation (20) to (22) shows how to compute the sum of the expected utility value:

$$\Sigma E(U(\pi(X_{ij}))) = \Sigma\Sigma E(U(\pi(X_{ij} = 0))) + \Sigma E(U(\pi(X_{ij} = 1))) \tag{20}$$

$$\Sigma E(U(\pi(X_{ij}))) = \Sigma(\theta_{ij} * N_{ij} * P_{ij} - C_{ij}) * X_{ij} \tag{21}$$

$$\Sigma E(U(\pi(X_{ij}))) = \Sigma(\theta_{ij} * (a - b * P_{ij}) * P_{ij} - C_{ij}) * X_{ij} \tag{22}$$

Here, it examines two constraints: the maximum number of air routes allowed to be put into use K, and total fleet passenger capacity N.

Constraint 1: the maximum number of routes allowed to be put into use

$$0 \leq \sum_{ij} X_{ij} \leq K \tag{23}$$

Constraint 2: Passenger capacity

$$0 \leq \sum_{ij} X_{ij} * N_{ij} \leq N \tag{24}$$

It has the assumption for the relationship between demand and price level (equation (11)). Hence, the research has:

$$0 \leq \sum_{ij} X_{ij} * (a - b * P_{ij}) \leq N \tag{25}$$

The research will build the model with multiple pairs of departure and destinations in a network in this part. Combining the constraints and functions above, it constructs the model for the full network on n hubs. Same as the simplest case, it assumes that the air planner chooses to open the routes to maximize profit. This leads to the following optimization problems.

In the case of the maximum flight constraint, it shown in the equation (26):

$$\begin{aligned} &\max_{X} \sum_{i,j=1}^{n} E(U(\pi(X_{ij}))) \\ &s.t. \quad X_{ij} \in \{0,1\} \\ &\qquad \sum_{ij} X_{ij} \leq K \end{aligned} \tag{26}$$

If instead, the research imposes a maximum passenger capacity, it instead arrives at equation (27):

$$\begin{aligned} &\max_{X} \sum_{i,j=1}^{n} E(U(\pi(X_{ij}))) \\ &s.t. \quad X_{ij} \in \{0,1\} \\ &\qquad \sum_{ij} X_{ij}(a - bP_{ij}) \leq N \end{aligned} \tag{27}$$

If the research imposes both constraints on the model, it has the equation (28):

$$\begin{aligned} &\max_{X} \sum_{i,j=1}^{n} E(U(\pi(X_{ij}))) \\ &s.t. \quad X_{ij} \in \{0,1\} \\ &\qquad \sum_{ij} X_{ij} \leq K \\ &\qquad \sum_{ij} X_{ij}(a - bP_{ij}) \leq N \end{aligned} \tag{28}$$

Proof of Optimization

In this section, the research presents proof of the solution to the maximization optimization problems expressed in equation (26). Proof of equation (27) and equation (28) is beyond the scope of this research, and so the research does not provide further context on solving these problems. The optimization problem expressed in (26) is an integer problem, and therefore generally hard to solve efficiently. This section seeks to provide an explicit characterization of the optimal solution. This model considers the cost of setting up and operating the routes (C_{ij}) and the probability of success travel from i to j (θ_{ij}) as fixed constants.

For constraints, the study wants to mainly focus on the maximum number of air routes that are allowed to be put into use (K). So, it is:

$$\begin{aligned} &\max_{X} \sum_{i,j=1}^{n} E(U(\pi(X_{ij}))) = \Sigma(\theta_{ij} * (a - b * P_{ij}) * P_{ij} - C_{ij}) * X_{ij} \\ &s.t. \quad X_{ij} \in \{0,1\} \\ &\qquad \sum_{ij} X_{ij} \leq K \end{aligned} \tag{29}$$

Let $M_{ij} = \theta_{ij} * (a - b * P_{ij}) * P_{ij} - C_{ij})$
Therefore:

$$\max_X \sum_{i,j=1}^{n} E(U(\pi(X_{ij}))) = \Sigma(\theta_{ij} * (a - b * P_{ij}) * P_{ij} - C_{ij}) * X_{ij}$$

This is the same as:

$$\max_X \sum_{i,j=1}^{n} M_{ij} * X_{ij}$$

Then arrive at the constrained optimization problem in the networked setting, which is given by:

$$\begin{aligned} \max_{X_{ij} \in \{0,1\}} & \sum_{ij=1}^{n} M_{ij} X_{ij} \\ \text{s.t.} & \sum_{ij} X_{ij} \le K \end{aligned} \tag{30}$$

Theorem 1: Define $v \in R^{n(n-1)}$ such that v_k is the kth largest value in M. Then, it has $v_1 \ge v_2 \ge \ldots \ge v_{n(n-1)}$. Construct $y \in \{0,1\}^{n(n-1)}$ similarly by ordering the elements of X in the same way, such that y_k corresponds to the route whose expected profit is v_k. Now, the optimization problem becomes:

$$\begin{aligned} \max_{y \in \{0,1\}^{n(n-1)}} & \sum_k v_k y_k \\ s.t. & \sum_k y_k \le K \end{aligned} \tag{31}$$

Let y^* be the optimal value for this problem. If $v_k > 0$ and $v_1 > v_2 > \ldots > v_{n(n-1)}$ then $y * k = 1$ if and only if $k \le K$ and $v_k \ge 0$.

Proof: the research proves Theorem 1 by contradiction. Let $\bar{y} \ne y^*$ be feasible for the problem (31). The research will show that $\bar{y}$ cannot be optimal.

First, introduce some notation. Let e_k be the kth canonical basis vector, which is equal to 1 in the kth entry and 0 elsewhere.

The research will explore two cases in the proof. One is the case that $\Sigma_k \overline{y_k} = K$, and another is the $\Sigma_k \overline{y_k} < K$.Both cases originated from the constraints of the problem (28).

Consider the case where $\Sigma_k \overline{y_k} = K$, and the constraint holds with equality. In this case, there must exist $\ell, m \in [1, \ldots, n(n-1)]$ such that it has both0=$\overline{y_\ell} \ne y_\ell{}^* = 1$ and 1 =$\overline{y_m} \ne y_m{}^* = 0$.

It can construct another feasible solution to the problem (28) given by $\bar{y} + e_\ell - e_m$. This solution is feasible since it satisfies both constraints. The objective value of this new solution is given by

$$\sum_k v_k(\bar{y} + e_\ell - e_m) = \sum_k v_k \bar{y}_k + v_\ell - v_m$$

By the assumption on y^*, it knows that $\ell < m$. Therefore, since $v_\ell > v_m$, the objective value of this new solution is larger, and $\bar{y}$ cannot be optimal.

Then consider the case for $\Sigma_k \overline{y_k} < K$. In this case, there exists $\ell \in [1, \dots, K]$, such that it has $0 = \overline{y_\ell} \neq y_\ell{}^* = 1$. It can set up a feasible solution to the problem (28) by $\bar{y} + e_\ell$. The objective value of this new solution is given by

$$\sum_k v_k(\bar{y} + e_\ell) = \sum_k v_k \bar{y}_k + v_\ell$$

Recall the assumption that $y_k^* = 1$ if and only if $v_k > 0$ and $k \leq K$. Since it knows that $0 = \overline{y_\ell} \neq y_\ell{}^* = 1$, hence, it has $v_\ell > 0$. Therefore, this new solution has a larger objective function, thus $\bar{y}$ cannot be optimal.

From above, the research proves that any $\bar{y} \neq y^*$ cannot be optimal in both cases. Therefore, y^* must itself be optimal.

Discussion

This research focuses on air transport network optimization. In the study, we make several assumptions to simulate the process of deciding to construct new air routes in a network for the air transport industry. We study the expected utility of profit under a linear demand function to build the baseline model and characterize the optimal solution. Then, from the baseline model, we develop an advanced model with a whole network. We explore the optimization results of the situation by constructing and analyzing the baseline model and the model with the network. We verify the result developed from the model to be the optimal result under the situation by mathematics analysis.

Here, we provide some potential applications and research directions related to this study. The model developed is about a point-to-point network structure with two hubs on one route. But in the real world, the situation is much more complicated; demand may be stochastic. Consequently, we will discuss several complexities which could enhance the feasibility of the model to reality: passenger demand, flight connections, and temporal dynamics.

In our model, we assume that the relationship between demand and price level is linear. However, the situation may be more complex. If the air transport experiences a shock, the parameter of the function may be unstable and change. Moreover, if the air industry suffers a sudden severe shock, the relationship will completely change. For instance, during the COVID-19, the demand for air transport promptly shrank: the number of passengers decreased by 60 percent in 2020 [3]. The whole system was seriously affected. As a result, the assumption of linear demand function may not work in different situations. The relation- ship between demand and price level needs to present with a new approach considering the characteristics of the specific situation. Furthermore, due to the varied situation such as weather and policy, the demand of passengers may be stochastic. For example, there generally exists peak-season and off-season in the air transport, which causes uncertainty on the demand [6]. Considering the random demand, we may need to apply different methods to improve the model and obtain the optimization results – perhaps applying average values or using numerical simulation.

In the study, we consider the base case with two hubs on one route. However, in reality, sometimes because there is no direct flight connecting the departure and destination, passengers will choose to transfer from departure to a transit station then to the destination instead of traveling directly to the destination. Under this situation, the demand of the passenger who wants to travel on the route will be different, and the relationship between the demand and price level will be affected. In this case, the situation is different from our original model. For the demand of passengers, if there exists a transit station, they will have additional travel routes to arrive at the destination. As a consequence, the demand of passengers of the direct route from departure to the destination will usually decrease. Considering this, our model may have additional constraints and variables when we construct the optimization model. We could start from the simplest case to understand and analyze the model: there are three hubs in the network. In this case, we assume a situation that we have a departure, a transit station, and a destination in our model. The air company already has two air routes from the departure to the transit station and from the station to the destination. There is not a direct route between departure and destination. However, there is still an existing demand for traveling directly from departure to destination. So the air planner wants to make a decision about whether the air company has to open the routes from departure to the destination. Under the assumption, there are connections among the three routes: the connection in the decision variable; the connection in the demand of passengers on the three fleeting routes.

Similar to how we do not consider the limitations of passenger demand and flight connections, our model also does not incorporate temporal dynamics. In our research, the fleeting routes operating at different period will not be influ- enced by time-based events. For instance, the price level or the demand could be time-dependent, varying by time. Besides, the probability of the successful transmission could be influenced by the time effect: the probability of the next period may have a relationship with the probability of the last period. To in- volve the time effect in the model, we could apply some methods like dynamic programming to improve our model. Besides, the probability of a successful flight can be varied in a time period. For instance, during the COVID-19 pan- demics, the probability of successful flight is highly correlated to the timely epidemic situation. If the pandemics break out or there is a huge number of infectious cases on the flight departing from the region, the air company may tend to fuse the flight from the departure hub due to the epidemic prevention concern and policy. In this case, the probability of a successful flight may be influenced by time.

While the above considerations would greatly improve the value of research into air transport network design, this study provides a meaningful contribution towards guiding later experiments in this field. The model involves the parame- ters of the air transport route design, providing a method to develop the model. Besides, we apply the expectation function to handle the uncertainty, which could apply to network questions with uncertainty. Additionally, the baseline model analyzes the simplest case of the fleeting route design between two hubs and the relationship among the parameters, including the demand and price level. Thus the model acts as the initiation to understand the route design. The baseline model also shows how the air planner makes the decision based on the optimal result of the model. Moreover, we build the networked model to simulate the situation in multiple pairs of hubs, thus being more applicable in reality. Through the construction of a theoretical network framework, we provide future experimenters with a theoretical point from which to begin their exami- nation of actual networks. In addition, we verify the optimal results we obtain from the networked model to be optimal by math analysis, which indicates that our optimization is established under the situation. Future research can use our theoretical model to simulate the air transport network design, understand the structure of the air network, and facilitate stronger practical implementations of air route network design.

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