Contractible Edges and Peripheral Cycles in 3-Connected Graphs

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ABSTRACT

Peripheral cycles (induced non-separating cycles) in a general 3-connected graph are analogous to the faces of a polyhedron. Using the works of various authors, this paper explores the distribution of contractible edges in 3-connected graphs as needed to prove a major result originally by Tutte: each edge in a 3-connected graph is part of at least 2 peripheral cycles that share only the edge and its end vertices. A complete, alternative proof of this theorem is provided. The inductive step is generalized into a new independent lemma, which states that each edge in a 3-connected graph with a non-adjacent contractible edge has at least as many peripheral cycles as in the contracted one.

I. Introduction

The theory of 3-connected graphs was created by Tutte in 1961 [Tut61]. A graph is 3-connected if it remains connected after removing one or two vertices together with their adjacent edges (all precise definitions used in the paper are in section II). One crucial notion in this theory is the concept of a contractible edge: an edge in a 3-connected graph is contractible if the graph remains 3-connected after the edge's contraction. Contractible edges have been extensively studied by many authors; in particular, by Ando et al [AES87], Ota and Saito [OtS88], and Saito [Sai90] in their works concerning the distribution of contractible edges in 3-connected graphs, which are used in this paper.

Tutte [Tut61] introduced the notion of a peripheral cycle which generalizes to any 3-connected graph an analogue of the boundary of a region in a 3-connected planar graph (i.e. the graph of a polyhedron). This corresponds to the faces of a polyhedron. Tutte proved [Tut63] that a graph is planar iff for every edge, the graph has exactly two peripheral cycles containing it; in such a graph, the vertices of that edge are the only ones in common. He also proved that any edge in a general 3-connected graph has at least two peripheral cycles containing it. A goal here is to provide an alternative proof of Tutte's latter result.

Theorem (3.4). Let G be a 3-connected graph, and $e \in E(G)$. Then there are peripheral cycles C_1, C_2 in G such that $C_1 \cap C_2 = G[v(e)]$.

There were other proofs of this theorem devised by Thomassen [Tho80], Ota and Saito [OtS88], and Kelmans [Kel81]. The proof presented here is new and independently developed, though similar in essence to the one presented by Thomassen. One of the differences of this proof compared to the previous ones is the use of a new lemma that may have other future applications.

Lemma (3.1). Let G be a 3-connected graph. Let $e, f \in E(G)$ such that f is contractible and not adjacent to e. By the definition of graph contraction, $e \in E(G/f)$. Let $C_1, \ldots, C_k$ be peripheral cycles in G/f that contain e such that for all $i \neq j$, $C_i \cap C_j = G[v(e)]$. Then there are peripheral cycles $D_1, \ldots, D_k$ in G such that for all $i \neq j$, $D_i \cap D_j = G[v(e)]$ as well.

In addition to definitions, section II of this paper contains a basic lemma and results about the existence and pattern of distribution of contractible edges in 3-connected graphs, due to Tutte [Tut61], Ando et al [AES87], Halin [Hal69], and Saito [Sai90]. Section III contains the proof of lemma 3.1 and finishes with the proof of theorem 3.4. The presentation in this paper does not assume any prior knowledge of graph theory.

II. Notations, Definitions, and Results Used

When words are underlined in a sentence, they are being defined there. Most definitions are based on Ando et al 1987, Diestel 2005, Thomassen 1981, and Tutte 1961 [AES87][Die05][Tho80][Tut61].

Graphs and subgraphs. A graph G is a pair of disjoint sets (V, E) and a map $E \rightarrow V^2 / [(x,y) = (y,x)]$. For the graph G, an element x of V, denoted as $x \in V(G)$, is called a vertex, and an element e of E, denoted as $e \in E(G)$, is called an edge. A set of vertices alone can be identified as a graph with those vertices and no edges. The unordered pair of vertices (x,y) assigned to an edge e are the ends of the edge; we say that x and y are incident to e, denoted $x \in v(e)$. The set of edges in G of which the vertex x is an end will be denoted $E_G(x)$. The number of ends that the vertex x is part of is its degree, denoted $d_G(x)$. Two vertices are adjacent if there is an edge for which they are ends. Two edges are adjacent if they have the same end vertex. The set of vertices in G to which the vertex x is adjacent will be denoted $V_G(x)$. Two edges with both of the same ends are parallel. An edge where both ends are the same vertex is a loop. A simple graph cannot have parallel edges or loops (otherwise, the graph is a multigraph). All graphs discussed will be assumed to be simple graphs with $|V(G)|$ being finite. For a simple graph, an edge with ends x,y can be written xy. The empty graph has no edges or vertices. A graph H is a subgraph of a graph G, denoted $H \subset G$, if $V(H) \subset V(G)$ and $E(H) \subset E(G)$, with the same incidence relations as G. A vertex $x \in V(H)$ is a vertex of attachment if $E_H(x) \neq E_G(x)$. Given subgraphs A and B of G, define $E_G(A, B) = \{e \in E(G) \mid V(A) \cap v(e) \neq \varnothing , V(B) \cap v(e) \neq \varnothing \}$.

Operations on graphs. The union of subgraphs A & B (denoted $A \cup B$) is defined by $V(A \cup B) = V(A) \cup V(B)$ and $E(A \cup B) = E(A) \cup E(B)$. Intersection of subgraphs is defined analogously: $V(A \cap B) = V(A) \cap V(B)$ and $E(A \cap B) = E(A) \cap E(B)$. Given a set of vertices $U \subset V(G)$, the induced subgraph of G on U, denoted G[U], is defined by $V(G[U]) = U$ and $E(G[U]) = \{e \in E(G) \mid v(e) \subset U\}$. The subgraph $G - U$ created by removing a vertex set U from a graph G is defined by $V(G - U) = V(G) \setminus U$ and $E(G - U) = \{e \in E(G) \mid U \cap v(e) = \varnothing \}$. The subgraph $G - \{x\}$ can also be written as $G - x$. The subgraph $G - H$ of G created by removing a graph H from a graph G is defined as $G - V(G \cap H)$. The subgraph $G \setminus e$ created by removing an edge e from a graph G is defined by $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) \setminus \{e\}$. The graph G / e created from G by contracting an edge e (with ends x,y) into a new vertex v_e is defined by $V(G / e) = (V(G) \setminus \{x,y\}) \cup \{v_e\}$ and $E(G / e) = E(G - \{x,y\}) \cup \{e_f \mid f \in E(G) \setminus e, v(f) \cap \{x,y\} \neq \varnothing , v(e_f) = \{v_e\} \cup (v(f) \setminus \{x,y\})\}$.

Paths and cycles. A path H in G is a non-empty subgraph of G in which the edges can be formed into an ordered list such that each edge is adjacent to the next one in the list, and all vertices have degree 1 or 2 in H. Its length is $|E(H)|$, with $|S|$ denoting the number of elements in a set S. If two vertices x,y have degree 1 in the path H, they are its ends; H is a path from x to y, also called an x-y path. Otherwise, the path is a cycle and can be referred to as a path from x to x for any $x \in V(H)$. A k-cycle is a cycle of length k. For a cycle H in G, an edge in $G - E(H)$ which whose ends are in H is a chord. An induced cycle in G is an induced subgraph of G that is also a cycle; it has no chords.

Connectivity. A non-empty graph is connected if there is a path between each pair of distinct vertices. An induced k-cycle H, for $k > 2$, is a peripheral cycle if $G - H$ is connected. A set of vertices U for which $G - U$ is not connected is a cut set. A maximal subgraph with a certain property is a subgraph which is not contained in any other subgraph having that property. A graph G is k-connected for an integer $k > 1$ if $|V(G)| > k$ and the subgraph created by removing any k-1 vertices from G remains connected; i.e. the smallest cut set must have at least k vertices. Most of this paper will be concerned with 3-connected graphs. For a 3-connected graph G, if G/e is 3-connected, then e is a contractible edge in G.

Lemma 2.1. (Folklore). Suppose $H \subset G$ is connected. If, for any vertex x of G – H, there is a path in G from x to some vertex of H, then G is connected.

Proof. We need to show that there is a path connecting any two vertices of G.

Case I: Let $a,b \in V(H)$. Since H is connected, there is an a-b path in G.

Case II: Let $a \in V(H)$, $b \in V(G-H)$. By assumption, there is a b-v path P_0 in G for some $v \in V(H)$. Then there is $p \in V(P_0)$ such that p is the first vertex from b that is not in $V(G-H)$. Let P_1 be the b-p path that is a subgraph of P_0. Since H is connected, there is a p-a path P_2 in H. Since the p-a path $P_2 \subset H$, and $P_1 \cap H = \{p\}$, we see that $P_1 \cup P_2$ is an a-b path in G.

Case III: Let $a,b \in V(G-H)$. By assumption, there is an a-u path P_1 in G and a b-v path P_2 in G for some $u,v \in V(H)$. Also, since H is connected, there is a u-v path P_3 in H. We now need to find an a-b path in $P_1 \cup P_2 \cup P_3$. Let $p_1 \in V(P_1)$ be the first vertex from a that is also in P_3. Let Q_1 be the a-p_1 path that is a subgraph of P_1, and let Q_2 be the p_1-v path that is a subgraph of P_3. We see that $Q_1 \cup Q_2$ is an a-v path in G. Similarly, Let $p_2 \in V(Q_1 \cup Q_2)$ be the first vertex from a that is also in P_2. Let Q_3 be the a-p_2 path that is a subgraph of $Q_1 \cup Q_2$, and let Q_4 be the p_2-b path that is a subgraph of P_2. We see that $Q_3 \cup Q_4$ is an a-b path in G.

In either case, there is a path connecting any two vertices of G, so G is connected. □

Theorem 2.2 [Tut61]. Let G be a 3-connected graph with $|V(G)| \geq 5$. Then G has a contractible edge.

A excellent simple proof of this theorem is provided by Robin Thomas [ThoR].

Theorem 2.3. [Sai90] Let G be a 3-connected graph with $|V(G)| \geq 6$. Suppose $X \subset V(G)$ such that every contractible edge of G has an end in X. Then $|X| > 2$.

In other words, the set of contractible edges of G needs more than 2 vertices to cover it.

Corollary 2.4. Let G be a 3-connected graph with $|V(G)| \geq 6$, and let $e \in E(G)$. Then there is a contractible edge $f \in E(G)$ that is not adjacent to e.

Proof. Suppose every contractible edge is adjacent to e. Then every contractible edge of G has an end in $v(e)$, and $|v(e)| = 2$. This immediately contradicts theorem 2.3. □

III. Peripheral cycles in 3-connected graphs

Lemma 3.1. Let G be a 3-connected graph. Let $e, f \in E(G)$ such that f is contractible and not adjacent to e. By the definition of graph contraction, $e \in E(G/f)$. Let $C_1, \dots, C_k$ be peripheral cycles in G/f that contain e, such that for all $i \neq j$, $C_i \cap C_j = G[v(e)]$. Then there are peripheral cycles $D_1, \dots, D_k$ in G such that for all $i \neq j$, $D_i \cap D_j = G[v(e)]$ as well.

Proof. Let $v(f) = \{u, v\}$, and let $w \in V(G/f)$ be the vertex created by contracting f. By definition, $e \in E(G - \{u, v\})$ and $G - \{u, v\} = (G/f) - w$. In particular, since $e \in E(C_i)$, we have $e \in E(C_i - w)$. Let C be one of the cycles

$C_1, \dots, C_k$. We will construct a peripheral cycle D in G such that C – w = D – {u, v}. The construction depends on whether $w \in V(C)$, and if so, it further depends on the local geometry.

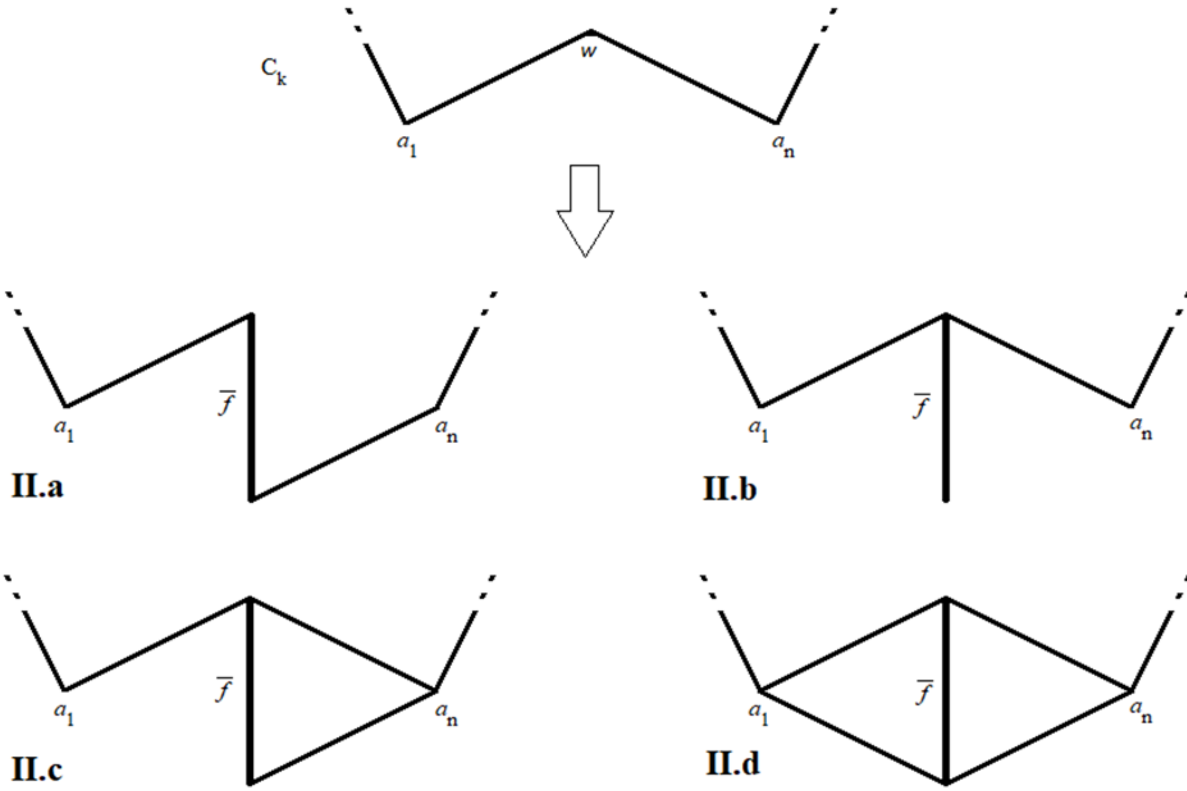

Figure 1

Case I: $w \notin V(C)$

Since $C \subset G / f - w = G - \{u, v\}$, C is also an induced cycle in G, not just in G / f. We know that (G / f) – C is connected and $w \in (G / f) - C$, so there is $p \in V((G / f) - C) \cap V_G(w)$, and so $p \in V(G)$. By the definition of graph contraction, p must be adjacent to at least one of {u, v} in G, which are adjacent to each other. There is a 2-edge path in G connecting both u and v via p to G – C – {u, v} = (G / f) – C , which is connected. By Lemma 2.1, G – C is connected, thus C is a peripheral cycle of G. For this case, our constructed cycle D is exactly C .

Case II: $w \in V(C)$

Let $E(C) = \{wa_1, a_1a_2, \dots, a_{n-1}a_n, a_nw\}$. Then C – w is an a_1-a_n path in G – {u, v}. Let $S_{1u} = \{d \in E(G) \mid v(d) = \{a_1, u\}\}$, $S_{1v} = \{d \in E(G) \mid v(d) = \{a_1, v\}\}$, $S_{nu} = \{d \in E(G) \mid v(d) = \{a_n, u\}\}$, and $S_{nv} = \{d \in E(G) \mid v(d) = \{a_n, v\}\}$. Each of these sets can contain at most one element, because G is assumed to be a simple graph. Since $wa_1 \in E(G / f)$, at least one of S_{1u} or S_{1v} is not empty, and since $wa_n \in E(G / f)$, at least one of S_{nu} or S_{nv} is not empty. As a result, $2 \le |S_{1u} \cup S_{1v} \cup S_{nu} \cup S_{nv}| \le 4$. There are 4 possibilities for the local geometry in G due to the content of sets S_{1u}, S_{1v}, S_{nu}, S_{nv}; we will consider these sub-cases separately. Figure 1 shows these possibilities, and figure 2 identifies the corresponding constructed cycle D.

Case II.a.: $S_{1u} = S_{nv} = \varnothing$ or $S_{1v} = S_{nu} = \varnothing$.

Without loss of generality, assume $S_{1u} = S_{nv} = \varnothing$. Define D = G[V(C – w) ∪ {u, v}]. We see that $E(D) = \{uv, va_1, a_1a_2, \dots, a_{n-1}a_n, a_nu\}$, so D is an induced cycle. Also, G – D = G – {u, v} – V(C – w) = (G / f) – w – V(C – w) = (G /

f) – V(C), which is connected; D is a peripheral cycle.

Case II.b.: $S_{1u} = S_{nu} = \varnothing$ or $S_{1v} = S_{nv} = \varnothing$.

Case II.c.: $|S_{1u} \cup S_{1v} \cup S_{nu} \cup S_{nv}| = 3$; one of $S_{1u}, S_{1v}, S_{nu}, S_{nv}$ is empty.

The argument is the same in both cases; in either one, without loss of generality, we can assume $S_{1u} = \varnothing$. Define D = G[V(C – w) ∪ {v}]. We see that E(D) = {$va_1, a_1a_2, \dots , a_{n-1}a_n, a_nv$}, so D is an induced cycle. Since G is 3-connected, G – {v, a_n} is connected, and so there must be a vertex $p \in V_G(u) \setminus \{v, a_n\}$. Suppose $p \in$ V(D). Since $S_{1u} = \varnothing$, $p \neq a_1$. Let $p = a_x$ for some $1 < x < n$. Then, by the definition of graph contraction, $wa_x \in$ E(G / f), so $wa_x \in$ E(G[V(C)]) = E(C). But w in the cycle C is already adjacent to a_1 and a_n ; contradiction. Therefore, $p \notin$ V(D). Now, G – D – u = G – {u, v} – V(C – w) = (G / f) – w – V(C – w) = (G / f) – V(C), which is connected. Since G – D – u is connected, $p \in$ G – D – u, and $up \in$ E(G), by Lemma 2.1, G – D is also connected; D is a peripheral cycle.

Case II.d.: $|S_{1u} \cup S_{1v} \cup S_{nu} \cup S_{nv}| = 4$.

Since G / f is 3-connected, G / f – {a_1 , a_n} is connected, and so there must be a vertex $p \in V_{G/f}(w) \setminus \{a_1, a_n\}$. Suppose $p \in$ V(C), i.e. $p = a_x$ for some $1 < x < n$. Then, $wa_x \in$ E(G / f), so $wa_x \in$ E(G[V(C)]) = E(C). But w in the cycle C is already adjacent to a_1 and a_n ; contradiction. So $p \notin$ V(C). By the definition of graph contraction, since $wp \in$ E(G / f), there is an edge $d \in$ E(G) such that $p \in$ v(d) and v(d) ∩ {u,v} ≠ ∅. Without loss of generality, assume $d = up$. Define D = G[V(C – w) ∪ {v}]. We see that E(D) = {$va_1, a_1a_2, \dots , a_{n-1}a_n, a_nv$}, so D is an induced cycle. Since $p \notin$ V(C) ∪ {u, v}, $p \notin$ V(D). Now, G – D – u = G – {u, v} – V(C – w) = (G / f) – w – V(C – w) = (G / f) – V(C), which is connected. Since G – D – u is connected, $p \in$ G – D – u, and $up \in$ E(G), by Lemma 2.1, G – D is also connected; D is a peripheral cycle.

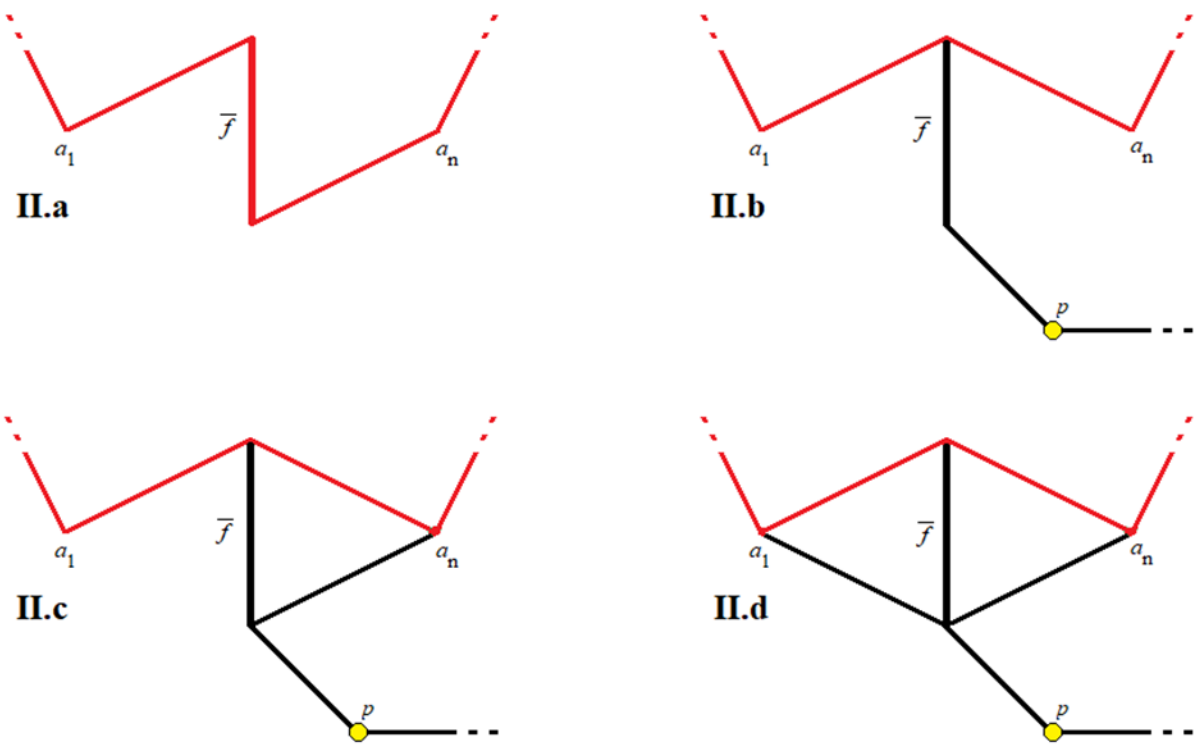

Figure 2

For each C_i, we construct D_i according to the above procedure. Since for all $i \neq j$, $C_i \cap C_j$ = G[v(e)] and $w \notin$ v(e), only one of $C_1, \dots , C_k$ can possibly contain w. If one exists, identify this cycle C_x . For all $i \neq x$, C_i falls into case

I, so $C_i = D_i$. We also have $C_x - w = D_x - \{u, v\}$, with $w \in V(C_x)$ and $\{u, v\} \cap V(D_x) \neq \varnothing$. No other C_i can contain w, and no other D_i can contain either u or v, so for all $i \neq x$, $C_i \cap C_x = C_i \cap (C_x - w) = D_i \cap (D_x - \{u, v\}) = D_i \cap D_x$. Furthermore, for all $i, j \neq x$, $C_i \cap C_j = D_i \cap D_j$. Therefore, for all $i \neq j$, $D_i \cap D_j = C_i \cap C_j = G[v(e)]$. ☐

The proof can be modified to show that if $C_1, \dots, C_k$ are merely distinct peripheral cycles in G / f that contain e, then there are distinct peripheral cycles $D_1, \dots, D_k$ in G that contain e. We will make the statement and proof explicitly.

<u>Lemma 3.2</u>. Let G be a 3-connected graph. Let $e, f \in E(G)$ such that f is contractible and not adjacent to e. By the definition of graph contraction, $e \in E(G / f)$. Let $C_1, \dots, C_k$ be distinct peripheral cycles in G / f that contain e. Then there are distinct peripheral cycles $D_1, \dots, D_k$ in G as well.

Proof. Let $v(f) = \{u, v\}$, and let $w \in V(G / f)$ be the vertex created by contracting f. By definition, $e \in E(G - \{u, v\})$ and $G - \{u, v\} = (G / f) - w$. In particular, since $e \in E(C_i)$, we have $e \in E(C_i - w)$. Let C be one of $C_1, \dots, C_k$. We will construct a peripheral cycle D in G such that $C - w = D - \{u, v\}$. As in Lemma 3.1, the construction depends on whether $w \in V(C)$, and if so, it further depends on the local geometry.

<u>Case I</u>: $w \notin V(C)$ As with Lemma 3.1, our constructed cycle D is exactly C, with the same reasoning.
<u>Case II</u>: $w \in V(C)$ Unlike in Lemma 3.1, there can be more than one cycle in which this holds. However, the construction of D remains the same, along with the proof that it is a peripheral cycle.

Suppose that for some $i \neq j$, we have $D_i = D_j$, despite $C_i \neq C_j$. Since $C_i \neq C_j$, there is $x \in V(C_i)$ such that $x \notin V(C_j)$. We have $C_i - w = D_i - \{u, v\} = D_j - \{u, v\} = C_j - w$. Suppose $x \neq w$. Then $x \in V(C_j)$; contradiction. So $x = w$, giving us $w \in V(C_i)$ and $w \notin V(C_j)$. By our construction of D_i and D_j, D_i contains at least one of $\{u, v\}$ while D_j contains neither; this contradicts $D_i = D_j$. Therefore, all the D_i are distinct. ☐

<u>Lemma 3.3</u>. Let G be a 3-connected graph on 4 or 5 vertices, and $e \in E(G)$. Then there are peripheral cycles C_1, C_2 in G such that $C_1 \cap C_2 = G[v(e)]$.

Proof. I. $n = 4$. The only 3-connected graph on 4 vertices is the tetrahedron K_4. By the known properties of this graph, every edge e is part of exactly 2 peripheral cycles C_1, C_2 of length 3. The third vertex is different for each cycle, so $C_1 \cap C_2 = G[v(e)]$.

II. $n = 5$. This case can be done by brute force checking of every graph on 5 vertices, but a more elegant argument is given here. By theorem 2.2, G has a contractible edge f. After contraction, G / f must be 3-connected and have 4 vertices; the only such graph is the tetrahedron K_4. Therefore, any 3-connected graph on 5 vertices can be obtained by splitting a vertex of a tetrahedron. Due to symmetry, any vertex can be used. This vertex has degree 3, so the vertices of the corresponding edge in G can have at most degree 4, and they must have at least 3, because G is 3-connected. So there are 3 options for their degrees: 3 & 3, 3 & 4, or 4 & 4. These correspond respectively to the graphs of the square pyramid, triangular hexahedron (bipyramid), and K_5 (the graph on 5 vertices with all edges). For the first two, there exist exactly 2 peripheral cycles for every edge, and they share only the vertices of that edge. For the last, there exist exactly 3 peripheral cycles for every edge, and they share only the vertices of that edge.

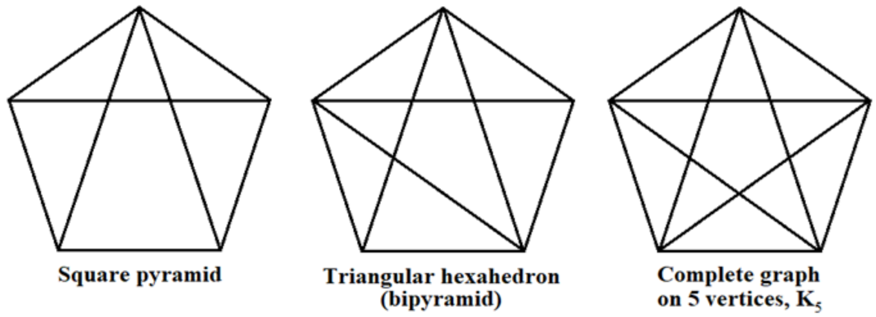

Figure 3

Theorem 3.4. [Tut63] Let G be a 3-connected graph, and $e \in E(G)$. Then there are peripheral cycles C_1, C_2 in G such that $C_1 \cap C_2 = G[v(e)]$.

Proof. By contradiction. Suppose the theorem is false. Then, let G be a graph on the smallest number of vertices for which it is false, and let $n = |V(G)|$. Since G is 3-connected, $n \geq 4$. By lemma 3.3, n cannot be 4 or 5, so $n \geq 6$. By corollary 2.4, there is a contractible edge $f \in E(G)$ that is not adjacent to e. Let $v(f) = \{u, v\}$. Since $e \in E(G / f)$ and $|V(G / f)| = n-1 < n$, there are peripheral cycles C_1, C_2 in G / f such that $C_1 \cap C_2 = G[v(e)]$. By lemma 3.1, there are peripheral cycles D_1, D_2 in G such that $D_1 \cap D_2 = C_1 \cap C_2 = G[v(e)]$. □

IV. Conclusions

Lemma 3.1 and 3.2 are new results. Lemma 3.1 is used to provide a novel, conceptually simplified proof of a well-known theorem by Tutte (Theorem 3.4). Since Lemma 3.1 is stronger than necessary to prove Tutte's theorem, it stands to reason that some extension of Theorem 3.4 could be proven using Lemma 3.1. Furthermore, Kriesell [Kri00] provides possible tools to do this.

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