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Simulating Charged Particles in Electromagnetic Fields

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ABSTRACT

Relativistic charged particles in electromagnetic fields follow paths that increase in complexity with an increasingly complex field. When traveling in these fields, the particles are acted upon by an electromagnetic force comprised of an electric component and a magnetic component. While solving for these paths in simple electromagnetic fields can be done analytically, the task becomes significantly more difficult when the fields get more complex. Thus, in these situations, numerical methods are required to find solutions. One of the most well-known such methods is the Boris method, which is explored in this paper. Before this method is applied to any complex situations, its accuracy must be ensured by testing on simple cases which are solvable by hand. These cases include a 1D electric field in the direction of the initial velocity of the particle, and a 1D magnetic field perpendicular to the particle's velocity. With the accuracy proven, the method was applied to the cases of a force-free field, a dipolar field, and a quadrupolar field. In the latter two cases the method produced very interesting results that could provide significant insight that would be very difficult to achieve analytically. In the case of the force-free field, however, the method shows some limitations, as a precise cancellation of the force produced by the electric and magnetic fields is required to produce a straight line and the Boris method has some difficulty achieving this, especially when using a large time step.

Introduction

Electromagnetic (EM) waves and fields appear all around us in the world. Whether it is the WiFi our phone uses to connect to the internet, or the X-rays that are used in hospitals, electromagnetism allows our world to run. In this manner, understanding how electromagnetic fields work and how they interact with charged particles becomes paramount to the success of our modern technology. These electromagnetic fields were concisely described by James Maxwell when he published Maxwell's Equations, 4 independent equations which described the behavior of electric and magnetic fields and how they relate to each other.
In differential form, Maxwell's Equations are given by [1]:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Where E ⃗ and B ⃗ are the electric and magnetic fields respectively, ρ is the charge density, J ⃗ is the current density, ε_0 is the vacuum permittivity, and μ_0 is the vacuum permeability.

The first law, also known as Gauss' Law, relates the electric field to the charge that produces that field, stating that the divergence of the electric field is proportional to the charge density in any given region. The second law states that the divergence of a magnetic field is always 0, meaning that all magnetic fields must be closed loops. Therefore, Maxwell's second law shows that there are no magnetic monopoles. Maxwell's third law (Faraday's law of induction) indicates that a time-varying magnetic field induces an electromotive force/voltage, which in turn produces an electric field. Thus, this law shows the unity between electric and magnetic fields. The fourth law has an interesting history with it initially being discovered by Ampere in the form:

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

In this form, Ampere's law stated that circulating magnetic fields are produced by electric currents. Maxwell, however, realized this law was incomplete and that he needed to add a new term:

$$\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

Maxwell's addition accounts for time varying electric fields, leading to the concept of displacement current. Maxwell's change allows the equation to be consistent with the conservation of charge, along with predicting the existence of electromagnetic waves that propagate in vacuum with a speed $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$, the speed of light.

For a while it was thought that these light waves must be oscillations of some substance which fills all of space. This was dubbed the aether. The idea was that Maxwell's equations only hold in the frame in which the aether is at rest; light should then travel at speed c relative to the aether. We now know that the concept of the aether is unnecessary baggage. Instead, Maxwell's equations hold in all inertial frames and are the first equations of physics which are consistent with the laws of special relativity. Ultimately, it was by studying the Maxwell equations that Lorentz was able to determine the form of the Lorentz transformations which subsequently laid the foundation for Einstein's vision of space and time.

To derive the equations that describe the motion of a charged particle in an electromagnetic field we begin with Newton's second law of motion:

$$\frac{d\vec{x}}{dt} = \vec{v}$$
$$m\frac{d\vec{v}}{dt} = \vec{F}(\vec{x}, \vec{v}, t)$$

However for speeds close to the speed of light relativistic effects need to be considered. In special relativity, the concept of four-velocity u is introduced, with the spatial component defined as:

$$\vec{u} = \gamma\vec{v} = \gamma\frac{d\vec{x}}{dt}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the Lorentz factor. In our case, the force, $\vec{F}(\vec{x}, \vec{v}, \vec{t})$, is the electromagnetic force, which has two components: the electric force $\overrightarrow{F_E}$, which is described by Coulomb's law as $\overrightarrow{F_E} = q\vec{E}(\vec{x})$ and the magnetic force, $\overrightarrow{F_M}$, which is given by $\overrightarrow{F_M} = q\left(\vec{v} \times \vec{B}(\vec{x})\right)$. Thus, the electromagnetic force is given by:

$$\overrightarrow{F_{EM}} = \overrightarrow{F_E} + \overrightarrow{F_M} = q\left(\vec{E}(\vec{x}) + \vec{v} \times \vec{B}(\vec{x})\right)$$

Where q is the charge of the particle, $\vec{E}(\vec{x})$ and $\vec{B}(\vec{x})$are the electric and magnetic field vectors at a specific position $\vec{x}$, and $\vec{v}$ is the velocity of the particle. The relativistic equations of motion for a charged particle with rest mass m are then given by:

$$\frac{d\vec{x}}{dt} = \frac{\vec{u}}{\gamma}$$

$$m\frac{d\vec{u}}{dt} = q\left(\vec{E}(\vec{x}) + \frac{\vec{u}}{\gamma c} \times \vec{B}(\vec{x})\right)$$

Given these two equations, we can solve for both the four-momentum of the particle, as well as its position. Whereas for some simple cases the above equations can be solved exactly, for more complex field configurations there are no known analytical solutions and we have to resort to numerical methods to solve them. A very popular and efficient method to model relativistic particles in electromagnetic fields is called the Leapfrog method, specifically the Boris Leapfrog method. This method effectively captures the motion of the particles to a high degree of precision, allowing for accurate conclusions to be drawn about how charged particles move in different electromagnetic fields.

In this research paper, the Boris method will be presented, along with a couple of very simple test cases to prove the accuracy of the method. Then, this method will be used to model complex magnetic fields that occur near blackholes and neutron stars, allowing us to gain insight about how particles move in these extreme conditions.

Methodology

The Need for a Numerical Method

Computers are unable to calculate derivatives the same way we do. Instead, they use numerical methods to simplify these calculations into additions, subtractions, multiplications, and divisions. Thus, our first task in building a numerical model to approximate charged particle paths should be to change these equations into a form that the computer can handle. Furthermore, we can simplify these numerical methods by setting certain constants, such as c = 1, and normalizing all velocities around this scale. Thus, we change equations and apply these simplifications to obtain:

$$\frac{\mathrm{m}(\mathrm{u_{new}} - \mathrm{u_{old}})}{\Delta \mathrm{t}} = \mathrm{F_{old}}$$

$$\frac{\mathrm{x_{new}} - \mathrm{x_{old}}}{\Delta \mathrm{t}} = \frac{\mathrm{u_{old}}}{\gamma}$$

Generally, we know the position and momentum at the current timestep, $\overrightarrow{\mathrm{x^n}}$ and $\overrightarrow{\mathrm{u^n}}$, respectively. With this information we then want to find the position and momentum at the new timestep n+1, $\overrightarrow{\mathrm{x^{n+1}}}$and $\overrightarrow{\mathrm{u^{n+1}}}$.

The Leapfrog Method

To accomplish this, we use the Leapfrog method. The Leapfrog method [2] is the basis of the numerical model studied in this paper. This method is a numerical integrator known for its ability to solve second order differential equations to a high degree of accuracy. Compared to other methods, the leapfrog method has its own specific advantages. For one, it solves second order differential equations directly, whereas other solvers split them into multiple first order differential equations. Secondly, the leapfrog method is reversible. If we start from an initial position and advance time forward to get to a final position, say point A, we can set point A as the new position in the leapfrog method and reverse the direction of time and we'll end up back at the position we started from. While this aspect of the leapfrog method was not used in this research, it could definitely be used in practice. For example, if we know the position of a charged particle that is in a strong electromagnetic field and we want to know where that particle came from, we can use the leapfrog method in reverse to figure this out. The third and most important quality of the leapfrog method is that it approximately conserves energy, meaning that the leapfrog method could out-perform another method.

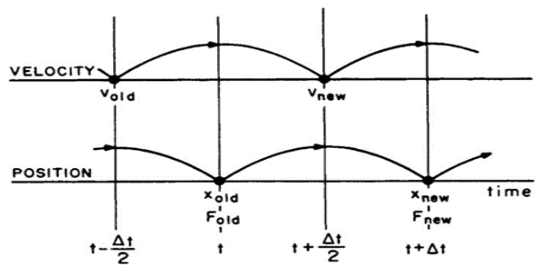

Figure 1. Leapfrog position and velocity update.

The idea of the leapfrog method can be seen in Fig 1. Both the position and the momentum are evaluated with the same timestep Δt, however they are staggered, with the momentum centered on integer timesteps and the position centered on half timesteps. In equation form, this looks like [3]:

$$\frac{\overrightarrow{u^{n+1}} - \overrightarrow{u^{n}}}{\Delta t} = \frac{q}{m}\left[\vec{E}\left(\overrightarrow{x^{n+1/2}}\right) + \vec{\bar{v}} \times \vec{B}\left(\overrightarrow{x^{n+1/2}}\right)\right]$$

$$\frac{\overrightarrow{x^{n+3/2}} - \overrightarrow{x^{n+1/2}}}{\Delta t} = \frac{\overrightarrow{u^{n+1}}}{\gamma}$$

The accuracy of the process can be increased by splitting the update of the position into two separate equations, using $x^n = \frac{x^{n+1/2}+x^{n-1/2}}{2}$. Applying this change would give the following series of equations:

$$\overrightarrow{x^{n+1/2}} = \overrightarrow{x^{n}} + \frac{\overrightarrow{u^{n}}}{2\gamma^{n}}\Delta t$$

$$\frac{\overrightarrow{u^{n+1}} - \overrightarrow{u^{n}}}{\Delta t} = \frac{q}{m}\left[\vec{E}\left(\overrightarrow{x^{n+1/2}}\right) + \vec{\bar{v}} \times \vec{B}\left(\overrightarrow{x^{n+1/2}}\right)\right]$$

$$\overrightarrow{x^{n+1}} = \overrightarrow{x^{n+1/2}} + \frac{\overrightarrow{u^{n+1}}}{2\gamma^{n+1}}\Delta t$$

This manner of computing the momentum and position over time leads to greater accuracy, however it also leads to some problems. The main issue is that to evaluate the position at the half-time step, we need the average velocity, $\vec{\bar{v}}$ at this half timestep. In turn, this average velocity requires the knowledge of the value of u^{n+1}. This is where the Boris method comes in.

The Boris Method

The Boris method [4] is just a specific way to define this average velocity to allow us to get rid of the problem. Specifically, the Boris method defines the $\vec{\bar{v}}$ as:

$$\vec{\bar{v}} = \frac{\overrightarrow{u^{n+1}} + \overrightarrow{u^n}}{2\gamma^{n+1/2}}$$

This change, thus, allows us to invert (reference equation 2) to get rid of the need of knowledge about the momentum at the n+1 timestep. This inversion results in the following equations:

$$\overrightarrow{u^-} = \overrightarrow{u^n} + \frac{q\Delta t}{2m}\vec{E}\left(\overrightarrow{x^{n+1/2}}\right)$$
$$\overrightarrow{u^+} = \overrightarrow{u^-} + \left(\overrightarrow{u^-} + \left(\overrightarrow{u^-} \times \vec{t}\right)\right) \times \vec{s}$$
$$\overrightarrow{u^{n+1}} = \overrightarrow{u^+} + \frac{q\Delta t}{2m}\vec{E}\left(\overrightarrow{x^{n+1/2}}\right)$$

In these equations, $\vec{t} = \vec{B}\left(\overrightarrow{x^{n+1/2}}\right) q\Delta t/(2m\gamma^-)$, $\vec{s} = 2\vec{t}/(1+t^2)$ and $\gamma^- = \sqrt{1+(\overrightarrow{u^-})^2}$. Furthermore, $\overrightarrow{u^-}$and $\overrightarrow{u^+}$ have their own symbolic meanings: $\overrightarrow{u^-}$ represents the first half of the acceleration on the particle by the electric field and $\overrightarrow{u^+}$ represents rotation of the particle due to the magnetic field. Thus, the full set of equations that must be solved are in order as follows:

$$\overrightarrow{x^{n+1/2}} = \overrightarrow{x^n} + \frac{\overrightarrow{u^n}}{2\gamma^n}\Delta t$$
$$\overrightarrow{u^-} = \overrightarrow{u^n} + \frac{q\Delta t}{2m}\vec{E}\left(\overrightarrow{x^{n+1/2}}\right)$$
$$\overrightarrow{u^+} = \overrightarrow{u^-} + \left(\overrightarrow{u^-} + \left(\overrightarrow{u^-} \times \vec{t}\right)\right) \times \vec{s}$$
$$\overrightarrow{u^{n+1}} = \overrightarrow{u^+} + \frac{q\Delta t}{2m}\vec{E}\left(\overrightarrow{x^{n+1/2}}\right)$$
$$\overrightarrow{x^{n+1}} = \overrightarrow{x^{n+1/2}} + \frac{\overrightarrow{u^{n+1}}}{2\gamma^{n+1}}\Delta t$$

These equations can now be translated into code and the numerical model is complete.

Proving Accuracy

As stated earlier, the equations of motion for a charged particle in an electromagnetic field become increasingly more difficult to solve as the electromagnetic field gets more complex. However, these equations are not very difficult to solve when the field is relatively simple. Thus, two simple cases will be explored: the first, a constant one-dimensional electric field and no magnetic field, and the second, a constant one dimensional magnetic field and no electric field. These two cases will first be solved analytically, then the results will be compared to the numerical solutions.

Constant 1D Electric Field

If we set $\vec{E} = (0,0,E_0)$, $\vec{B} = (0,0,0)$, $\vec{x} = (0,0,0)$, $\vec{v} = (0,0,0)$ the equations of motion become:

$$\frac{dz}{dt} = \frac{u_z}{\gamma}$$
$$\frac{du_z}{dt} = \frac{q}{m} E_0$$

Integrating the second equation to solve for u_z gives$u_z = \frac{q}{m} E_0 t$. Then substituting for u_z into the first equation we get$\frac{dz}{dt} = \frac{q}{m\gamma} E_0 t$. After integrating this first order differential equation, we are left with: $z(t) = \frac{mc^2}{qE_0}(\gamma(t) - 1)$, with $\gamma(t) = \sqrt{1 + \frac{(qE_0 t)^2}{(mc)^2}}$. For large t we have that $\gamma \approx \frac{qE_0 t}{mc}$.

For the numerical implementation we used $m=1$, $q=1$ and $c=1$. We chose a time step $\Delta t = 0.001$and ran the numerical simulation for 10000 time steps. We plot the theoretical and simulated γ as a function of time in Figure 2 and the relative error defined as $\frac{\gamma_{th} - \gamma_{boris}}{\gamma_{th}}$ in Figure 3. The analytical and numerical values agree very well.

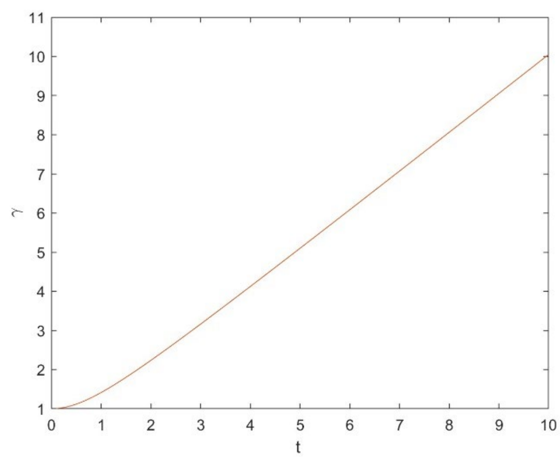

Figure 2. This figure illustrates γ as a function of time for a particle in a constant uniform electric field. The theoretical and numerical values are essentially identical.

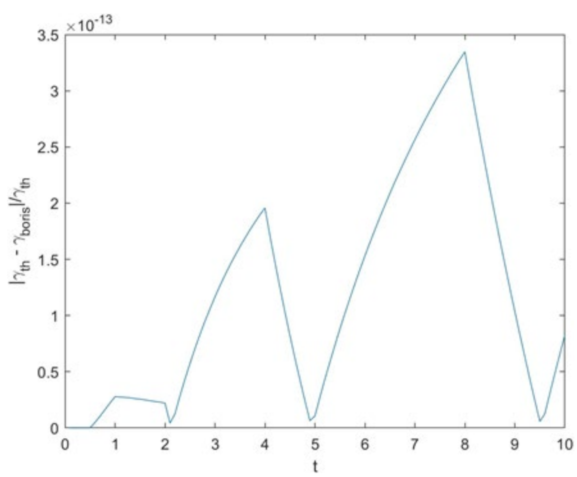

Figure 3. The relative error for γ as a function of time for a particle in a constant uniform electric field. The

accuracy of the numerical method is extremely good.

Constant 1D Magnetic Field

If we set $\vec{E} = (0,0,0)$, $\vec{B} = (0,0,B_0)$, $\vec{v} = (0, v_\perp, 0)$. The electromagnetic force in this example is given by the equation: $\overrightarrow{F_{EM}} = \frac{q}{m}\vec{v} \times \vec{B}$. This force acts perpendicular to the motion, meaning that it must be equal to the centripetal force the particle feels. Thus, we obtain the equation: $\frac{qv_\perp B_0}{c} = \frac{\gamma m v_\perp^2}{R_C}$. This demonstrates that in a constant, 1D magnetic field, the particles will orbit in a circular path with radius $R_C = \frac{\gamma m c v_\perp}{qB_0}$. For the numerical simulation we have again used $m = 1$, $q = 1$, $c = 1$ and $B_0 = 1$. The time step was chosen to be $0.001 \times T_{cyc}$ where $T_{cyc} = \frac{2\pi m}{qB}$ is the cyclotron period. We ran the simulation for 5 complete cycles. Again, the numerical solution matches very well the analytical solution as can be seen in Fig 4. Whereas the relative error in the gamma of the particle is quite small as shown in Fig 5 it seems to increase linearly with the simulation time. The relative error in the position of the particle is also fairly small as shown in Fig 6, but because of the small phase lag introduced by the Boris method [3] the particle does not complete a full circle in T_{cyc}.

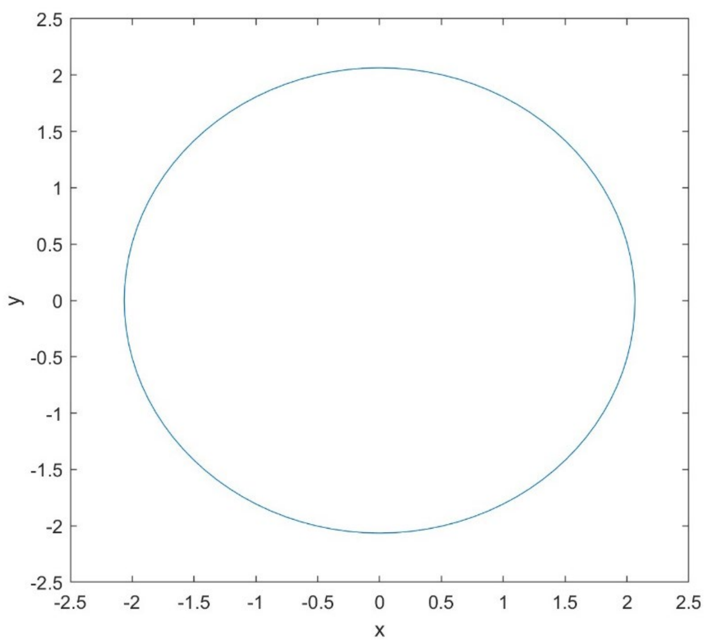

Figure 4. The path of a particle moving in a constant magnetic field in the z direction with initial velocity (0, 0.9, 0) is a circle with radius R_C.

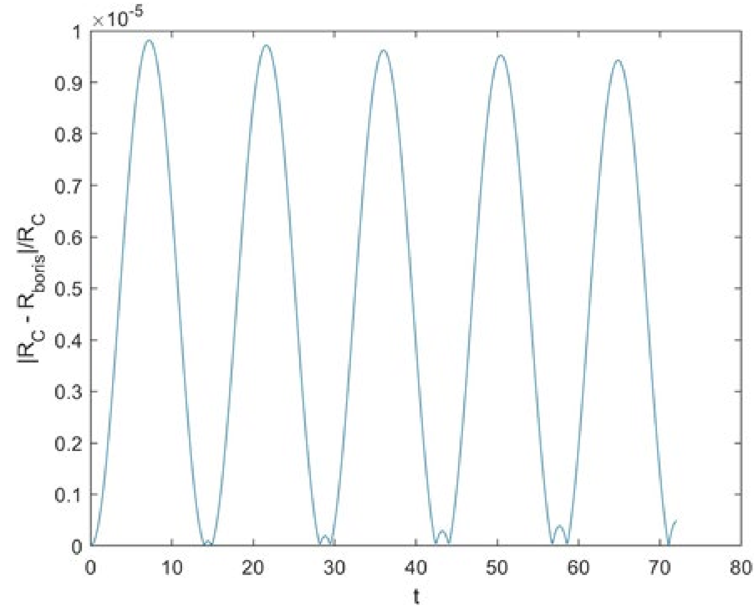

Figure 5. Relative error for the position of a particle moving in a constant magnetic field in the z direction with initial velocity (0, 0.9, 0). The error is quite small but there seems to be a phase shift and the particle does not complete a full circle in $T_{cyc.}$

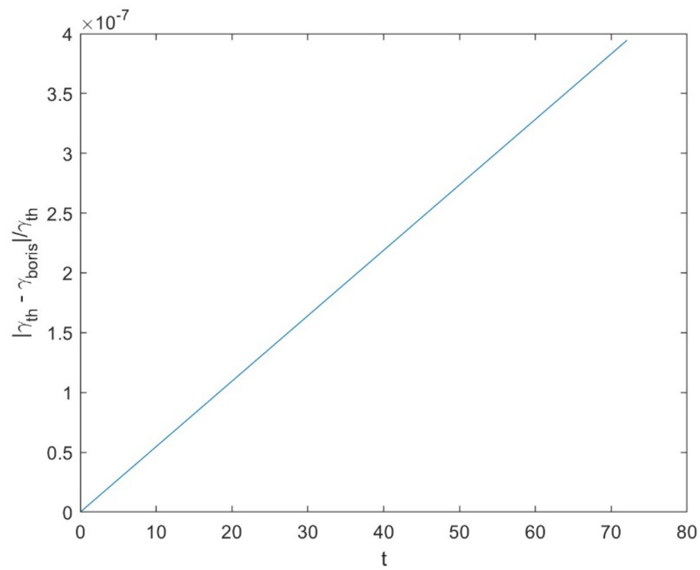

Figure 6. Relative error for the γ of a particle moving in a constant magnetic field in the z direction with initial velocity (0, 0.9, 0). Although very small, the error increases linearly with the simulation time.

Conclusion of Accuracy

For these two simple cases, the Boris pusher produced the expected results, giving identical paths to the analytical solutions. Therefore, the accuracy of the method is proved, meaning it can now be used to model more complex paths, allowing one to learn more about how particles interact with complex electromagnetic fields.

Test Cases

With the accuracy of the model for simple cases proven, it can now be used to model more complex and interesting electromagnetic fields. Allowing scientists to get both a qualitative and quantitative understanding of what happens in these more complex fields in our universe. The following electromagnetic fields were tested: a force-free field, a dipolar field and a quadrupolar field.

Force-Free Field

As the name suggests, all a force-free field is, is an electromagnetic field that exerts no net force. One way this can occur is if the field itself is 0 everywhere, however this isn't a very interesting case. A more interesting situation would be if the electromagnetic force was zero, but $\vec{E}$ and $\vec{B}$ were non-zero. In this case, we would have the equation: $\overrightarrow{F_{EM}} = q(\vec{E} + \vec{v} \times \vec{B}) = 0$. Thus, solving this equation for $\vec{E}$ gives $\vec{E} = -\vec{v} \times \vec{B}$.

When a particle is placed in a force free field with a constant velocity, it is expected that the particle will continue to move at this constant velocity since there is no net force acting on it. Thus, the resulting path of this field should be a straight line.

Dipolar Field

A dipolar field is the field produced by a magnet. As the name suggests, it is made up of two poles, a north pole and a south pole. Its magnetic field lines exit from the north pole of the magnet and re-enter at the south pole, forming a closed path. Dipolar fields are very common here on Earth, but also out in space. The primary example of dipolar fields in space are those around stars, massive neutron stars, and black holes. The dipolar fields around blackholes and neutron stars are the ones that the model is trying to represent, specifically, as those are the most powerful, meaning that the particles that orbit in them act the most relativistically. An ideal dipolar field has no azimuthal component, meaning that in spherical coordinates, it can be expressed as [3]:

$$\vec{B}(r,\theta) = \frac{M}{r_3}\left[2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}\right]$$

where r is the radial distance from the center of the dipole, $\hat{\theta}$ is the polar angle angle measured from the dipole axis, and M is the dipole moment. This magnetic field can be converted into Cartesian coordinates to produce the following field:

$$\vec{B}(x,y,z) = \frac{M}{(x^2 + y^2 + z^2)^{5/2}} \times [3zx\hat{x} + 3zy\hat{y} + (2z^2 - x^2 - y^2)\hat{z}]$$

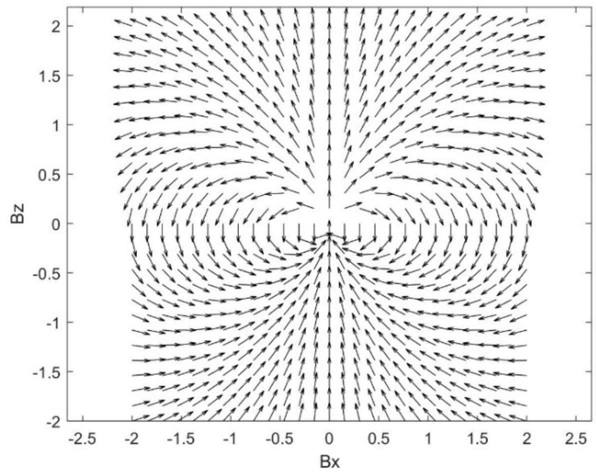

Figure 7. This figure gives a visual of the dipolar field. It represents a 2D slice of the 3D field at y=0. The graph is composed of vectors, pointing in the direction of the field. This view of the field allows for a qualitative analysis of its properties, allowing for the visualization of the symmetries expressed mathematically in the equations.

A visual representation of this dipolar field can be seen in Fig 7. For the numerical simulations used in this paper, the value of M is defined in a way such that $\frac{qM}{m} = 20$. Understanding how particles move in dipolar fields can be very useful as it can help scientists understand the nature of the stellar objects that form these fields, such as their mass and chemical composition.

Quadrupolar Field

A quadrupolar magnetic field is a complex field which can be created by four magnets arranged parallel to each other, where each magnet's north and south poles alternate. When an electric current flows through each magnet, it creates a magnetic field that alternates between repelling and attracting charged particles within a specific region. The resultant field is represented in Fig 8.

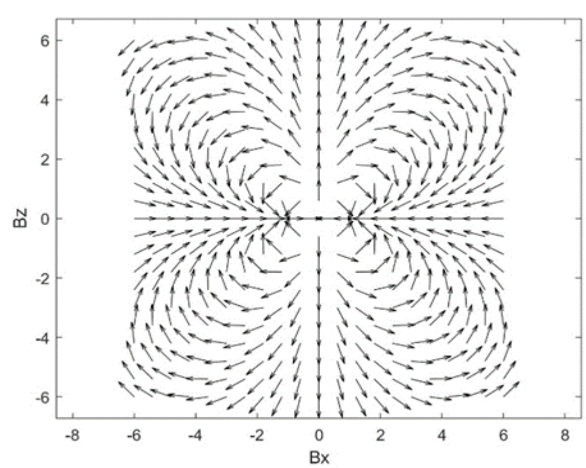

Figure 8. This figure gives a visual of the quadrupolar field. It represents a 2D slice of the 3D field at y=0. The qudrupolar field is obtained from 4 magnetic dipols placed on the x and z axis respectively, with alternating poles and separated by a small distance.

Results

In this section, the results from each one of the test-cases discussed above will be presented. Then, the produced paths will be both quantitatively and qualitatively analyzed to gain a deeper understanding of what occurs in each scenario.

Force-Free Field Results

As mentioned in the previous section, the force-free field occurs when $\overrightarrow{F_{EM}} = q(\vec{E} + \vec{v} \times \vec{B}) = 0$. Therefore, one would expect that a charged particle in this field should move with constant speed in a straight line. This is a very stringent test for the particle pusher, since a slight deviation from the exact cancellation of the net force causes errors in the solution [5]. To test this hypothesis we have chosen the magnetic field to be $\vec{B} = (0,0,1)$ and initial velocity (in units of c) $\vec{v} = (0,0.9,0)$. Therefore the electric field is $\vec{E} = (E_x, 0,0)$ where $E_x = -v_y B_z$. We have tested a number of time steps, ranging from $0.05 \times T_{cyc}$ to $0.001 \times T_{cyc}$ and ran the simulation for $100T_{cyc}$.

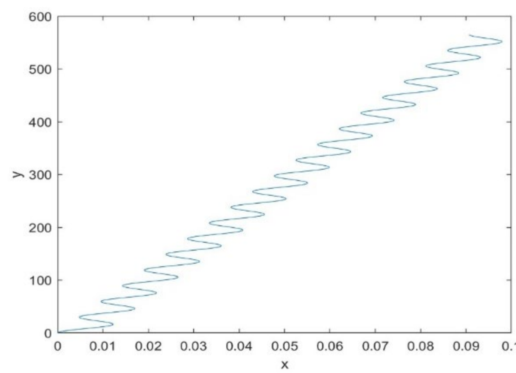

Figure 9. This figure illustrates the simulated trajectory of a charged particle in a force free electromagnetic field com- puted by the Boris method, using a time step of $0.05 \times T_{cyc}$. There is a drift in the x direction super-posed over an oscillatory motion.

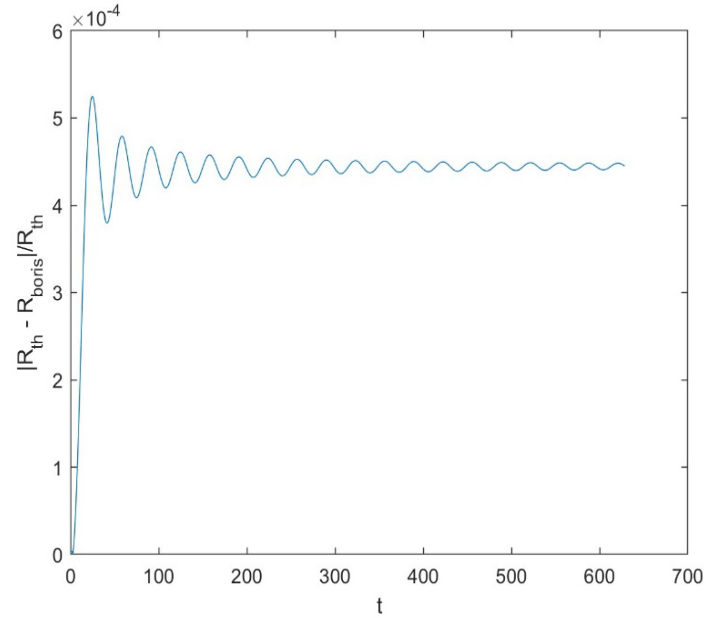

Figure 10. This figure illustrates the relative error of the distance traveled by a charged particle in a force free field, calculated using the Boris method and compared to the theoretical value. The relative error oscillates in the beginning then it seems to stabilize around 4.5×10^{-4}.

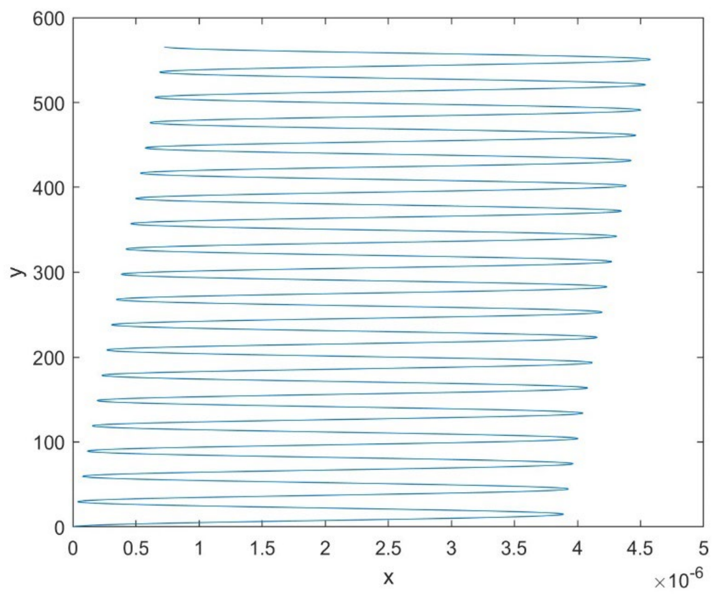

Figure 11. This figure illustrates the simulated trajectory of a charged particle in a force free electromagnetic field com- puted by the Boris method, using a time step of $0.001 \times T_{cyc}$. The drift and oscillation in the x direction is a lot smaller than for the $0.05 \times T_{cyc}$ time step.

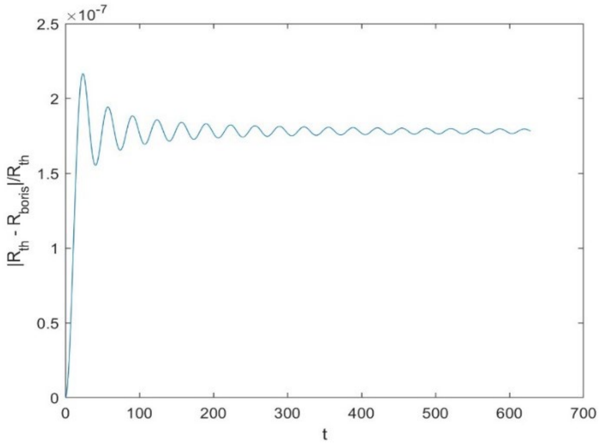

Figure 12. This figure illustrates the relative error of the distance traveled by a charged particle in a force free field, calculated using the Boris method and compared to the theoretical value. The relative error seems to stabilize around 1.75×10^{-7}, more than 3 orders of magnitude lower than the relative error for the $0.05 \times T_{cyc}$ time step.

The error in the x direction has an interesting behavior with an oscillatory motion superposed on a drift as can be seen in Fig 9 and Fig 11. These deviations from the expected position of the particle show a very strong dependence on the time step chosen, such that for a time step of $0.05 \times T_{cyc}$ the error is around 4.5×10^{-4} (Fig 10) whereas for a time step of $0.001 \times T_{cyc}$ the relative error is more than 3 orders of magnitude lower, around 1.75×10^{-7} (Fig 12).

Dipolar Field

To simulate the path of a charged particle in a dipolar field, a magnetic dipole along the z-axis and a particle with initial velocity $(0,0.2,0.5)$ placed at $(0.5,0.5,0)$ were used. A time step of $0.001 \times T_{cyc}$ was used and the simulation was ran for $5000 \times T_{cyc}$. The 3D path juxtaposed with the magnetic field can be seen in Fig 13. The particle seems to follow the field lines, changing direction when it gets very close to the north or the south poles of the magnet where the field lines become very dense as it can be seen in the detailed path of the particle shown in Fig 14.

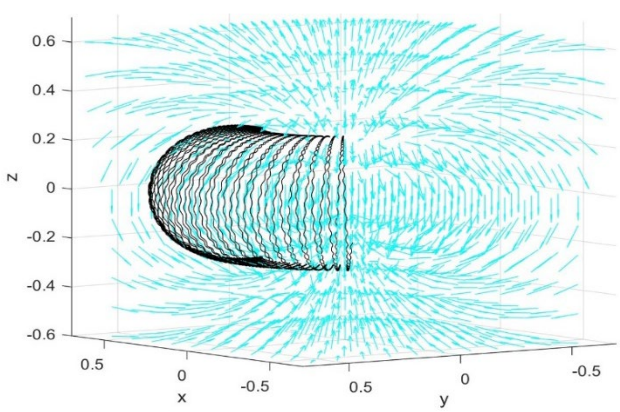

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Figure 13. 3D path taken by the particle (black) in a dipolar magnetic field (field lines seen in blue). The particle curls around the magnetic field lines, until it gets to a point in the field where the magnetic force is strong enough to turn it around.

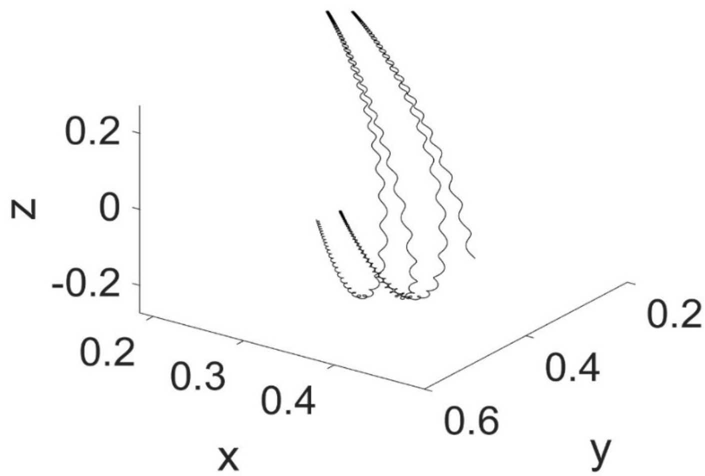

Figure 14. This figure illustrates the detailed 3D path of a particle in a dipolar magnetic field. The particle moves along the magnetic field lines, changing direction when it gets close to the north or the south poles of the magnet.

Quadrupolar Field

To simulate the path of a charged particle in a quadrupolar field 4 magnetic dipoles were used: 2 along the z-axis and 2 along the x-axis, separated by a distance of 1, with alternating north and south poles. Given this field configuration, a particle with a velocity along the y-axis should feel no net force, therefore it should move in a straight line. This hypothesis was tested, and this was indeed the case. However even a very small speed component along the x or z direction can get the particle almost trapped in the complex quadrupolar field. The detailed path of a particle with initial velocity $(0.1,0.5,0)$ starting at $(0,0,0)$ is shown in Fig 15.

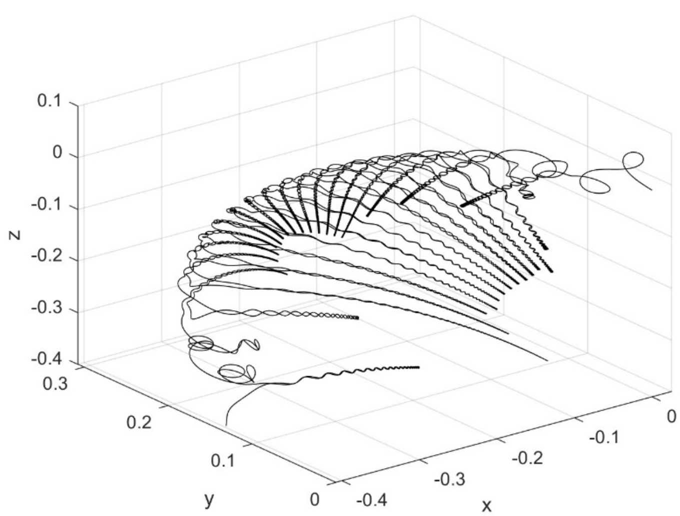

Figure 15. This figure illustrates the detailed 3D path of a particle with initial velocity (0.1, 0.5, 0) starting at (0, 0, 0) in a quadrupolar magnetic field. The particle follows a very complex trajectory and seems to be trapped by the quadrupolar field.

The rest of the settings are similar to the magnetic dipole simulation. As can be seen, the trajectory is very complex, almost chaotic and the particle seems to be trapped in the magnetic field for the duration of our simulation. Depending on the initial velocity used in simulation, some trajectories escape the center of the quadrupolar field and move away. However, for the simulation shown here the particle seems confined to a very small region near the center despite a large initial velocity along the y-axis, as it can be seen in Fig 16.

Discussion and Conclusions

The Boris method is one of the most widely used numerical methods to simulate charged particles in an electromagnetic field. We have shown that the method produces the expected behavior for the fields that can be solved analytically. For the force free field the Boris method shows some limitations, as a precise cancellation of the force produced by the electric and the magnetic fields is required to produce a straight line and the Boris method has some difficulty achieving this, especially when using a large time step.

The trajectories of charged particles in dipolar and quadripolar fields were subsequently simulated and provided us with interesting insights that would have been very difficult, if not impossible to achieve analytically.

The work done in this paper can be expanded in two main directions: explore more complex numerical methods that could behave better for example in the force free fields and simulate other complex magnetic fields, like a magnetic mirror for example, using the Boris method. Depending on the initial velocity used in simulation, some trajectories escape the center of the quadrupolar field and move away.

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