HIGH SCHOOL EDITION
Journal of Student Research

Volume 12 Issue 4 (2023)

Applications of the Spectral Theorem: Utilizing Eigenvalues and Eigenvectors

Maxwell Lachut[1] and Benjamin Keigwin[#]

[1]Christian Brothers Academy, USA
[#]Advisor

ABSTRACT

This paper explores the applications and principles of both the real and complex spectral theorems, cornerstones of linear algebra and functional analysis. We begin with an overview of the spectral theorem and delve into the pivotal roles of eigenvalues and eigenvectors, culminating in a detailed proof of the theorem. A discerning comparison is made between the real and complex versions; notably, the latter seamlessly integrates with the fundamental theorem of algebra, an attribute absent in the former. The ensuing sections illuminate practical applications. The real spectral theorem emerges as instrumental in multivariable calculus, notably in the second derivative test, in data-driven techniques such as Principal Component Analysis (PCA), and in electrical network analyses via Kirchhoff's matrix. The complex variant takes center stage in quantum mechanics, illuminating the Schrödinger Equation and underpinning the structure of Hilbert spaces. This paper underscores the spectral theorem's profound relevance in diverse theoretical and practical arenas.

Introduction

The Spectral Theorem, a fundamental theorem in the field of linear algebra, comprises two distinct but interconnected versions: the Real Spectral Theorem and the Complex Spectral Theorem. Each version holds unique applications across a broad spectrum of mathematical and scientific disciplines.

The Real Spectral Theorem posits that every real symmetric matrix can be diagonalized by a real orthogonal matrix, with the real eigenvalues situated along the main diagonal of the resulting matrix. Similarly, the Complex Spectral Theorem articulates that every complex self-adjoint matrix can be diagonalized by a unitary matrix, leading to a diagonal matrix with complex eigenvalues along the main diagonal. This diagonalization process is a pivotal characteristic of the Spectral Theorem.

Eigenvalues and eigenvectors are crucial to understanding and applying both versions of the Spectral Theorem. Eigenvalues, scalar values tied to a given matrix, indicate the factor by which the matrix stretches or compresses along the direction of its associated eigenvectors. For a square matrix A, an eigenvalue denoted by λ and its corresponding eigenvector v satisfy the following equation[1]:

$$Av = \lambda v$$

Under the Real Spectral Theorem, the eigenvalues are real numbers, and the corresponding eigenvectors are vectors in real space. In the process of diagonalizing a real symmetric matrix, these real eigenvalues populate the main diagonal of the resulting diagonal matrix.

Conversely, within the Complex Spectral Theorem, the eigenvalues are complex numbers and the eigenvectors are vectors in complex space. The diagonalization of a complex self-adjoint matrix places these complex eigenvalues along the main diagonal of the resulting matrix.

The roles of eigenvalues and eigenvectors in the Spectral Theorem underpin the diagonalization process, allowing for the transformation of matrices into more manageable forms. This transformation simplifies calculations and exposes key properties of the original matrix.

The Spectral Theorem remains a compelling tool in the study of linear algebra, offering deep insights into the diagonalization process of both real symmetric and complex self-adjoint matrices. Through the prism of eigenvalues and eigenvectors, the theorem enables the transformation of matrices into simplified forms, thereby facilitating their application in various fields. From the realms of physics to data analysis, the Spectral Theorem has proven invaluable in deciphering complex systems, solving differential equations, and unearthing inherent matrix structures.

In the subsequent sections, we will delve deeper into various applications of the Spectral Theorem. We will explore its impact on the formulation and solution of differential equations, discuss its significance in quantum mechanics and signal processing, and investigate its role in data analysis and machine learning. The aim of this paper is not only to provide a comprehensive overview of these applications, but also to demonstrate the elegance, versatility, and profound impact of the Spectral Theorem on modern mathematics.

Proof of Spectral Theorem

In this section, we will prove the Spectral Theorem for both real symmetric matrices (Real Spectral Theorem) and complex self-adjoint matrices (Complex Spectral Theorem).

Real Spectral Theorem

The Real Spectral Theorem states that a real symmetric matrix can be diagonalized by an orthogonal matrix. Let A be a real symmetric matrix.

Proof. Let's quickly discuss the key ideas that help us prove this (c.f. [1][2]):

1. The eigenvalues of A are real. If $Ax = \lambda x$, taking the transpose of both sides gives us $x^T A^T = \lambda x^T$. Since A is symmetric, $A^T = A$, so we get $x^T A = \lambda x^T$. Now taking the dot product of $x^T A$ with x and comparing it with $\lambda x^T \cdot x$ shows λ is real.
2. Eigenvectors corresponding to different eigenvalues of A are orthogonal. If λ_1 and λ_2 are distinct eigenvalues with eigenvectors x_1 and x_2, we find $(\lambda_1 - \lambda_2)x_1^T x_2 = 0$. Because λ_1 and λ_2 are distinct, $x_1^T x_2$ must be zero, meaning x_1 and x_2 are orthogonal.

These facts allow us to form a matrix P from A's normalized, orthogonal eigenvectors. Because its columns are orthogonal, P is an orthogonal matrix, which means $P^{-1} = P^T$.

When we compute $P^{-1}AP$, we obtain a diagonal matrix D with A's eigenvalues on its diagonal. Hence, $A = PDP^{-1}$ is the diagonalization of A by the orthogonal matrix P.

This concludes the proof that any real symmetric matrix A can be diagonalized by a real orthogonal matrix. □

Complex Spectral Theorem

The Complex Spectral Theorem asserts that a complex self-adjoint matrix can be diagonalized by a unitary matrix. Let us take A as a complex self-adjoint matrix.

Proof. To simplify, we will focus on two main aspects of A (c.f. [1][2]):

1. All eigenvalues of A are real. Consider an eigenvalue λ and corresponding eigenvector x, such that $Ax = \lambda x$. Taking the conjugate transpose of both sides gives $\overline{\lambda} x^H = x^H A^H$. Since A is self-adjoint, $A^H = A$, so we have $\overline{\lambda} x^H = x^H A$. Now taking the dot product of $x^H A$ with x and comparing it with $\overline{\lambda} x^H \cdot x$ shows λ is real.

2. Eigenvectors corresponding to different eigenvalues of A are orthogonal. If λ_1 and λ_2 are distinct eigenvalues with corresponding eigenvectors x_1 and x_2, we find $(\lambda_1 - \lambda_2)x_1^H x_2 = 0$. Because λ_1 and λ_2 are distinct, $x_1^H x_2$ must be zero, making x_1 and x_2 orthogonal.

Using these properties, we can construct a unitary matrix P from the normalized, orthogonal eigenvectors of A. P is unitary, so $P^{-1} = P^H$.

Computing the product $P^{-1}AP$, we obtain a diagonal matrix D with A's eigenvalues on its diagonal. Hence, $A = PDP^{-1}$ is the diagonalization of A by the unitary matrix P.

In conclusion, a complex self-adjoint matrix A can be diagonalized by a unitary matrix. □

Real vs. Complex Spectral Theorem

The distinction between the Real and Complex Spectral Theorems is highlighted by their respective interactions with the Fundamental Theorem of Algebra, which states that every non-constant polynomial with complex coefficients has at least one complex root. The implications of this theorem are profound when considering the existence of eigenvalues for matrices.

Real Spectral Theorem

The Real Spectral Theorem pertains specifically to real symmetric matrices[3]. These matrices are characterized by always having real eigenvalues, a property that is a direct consequence of their symmetric nature. Consider the generic symmetric matrix:

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Where $a, b, c \in \mathbb{R}$. The characteristic polynomial of A is real-coefficiented. Due to its symmetry, the eigenvalues of A are guaranteed to be real. Hence, such matrices are diagonalizable through real orthogonal transformations.

Examples

Example 1: Consider the real symmetric 3×3 matrix M:

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

We can diagonalize this matrix with the orthogonal matrix S and its inverse S^{-1} such that $M = SJS^{-1}$, where J is the diagonal matrix with the eigenvalues of M on its diagonal. The matrix S and its inverse are given by:

$$S = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} -1/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & -1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

And the diagonal matrix J is:

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

We have diagonalized M with the orthogonal matrix S, consistent with the Real Spectral Theorem.
Example 2: Consider the real symmetric 3×3 matrix M':

$$M' = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 7 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

We can diagonalize this matrix with the matrix S' and its inverse S'^{-1} such that $M' = S'J'S'^{-1}$, where J' is the diagonal matrix with the eigenvalues of M' on its diagonal. The matrix S' and its inverse are given by:

$$S' \approx \begin{pmatrix} -1.85362 & 0.654334 & 0.440669 \\ -0.173974 & -1.22369 & 1.05284 \\ 1 & 1 & 1 \end{pmatrix},$$
$$S'^{-1} \approx \begin{pmatrix} -0.415035 & -0.0389535 & 0.223905 \\ 0.22366 & -0.418274 & 0.341814 \\ 0.191374 & 0.457227 & 0.434282 \end{pmatrix}$$

And the diagonal matrix J' is approximately:

$$J' \approx \begin{pmatrix} -0.430739 & 0 & 0 \\ 0 & 1.84455 & 0 \\ 0 & 0 & 12.5862 \end{pmatrix}$$

We have diagonalized M' with the matrix S', consistent with the Real Spectral Theorem.

Complex Spectral Theorem

Contrastingly, the Complex Spectral Theorem focuses on complex self-adjoint (or Hermitian) matrices. Such matrices always yield a characteristic polynomial with complex coefficients. Given the Fundamental Theorem of Algebra, every such polynomial possesses at least one complex root, thereby ensuring the presence of eigenvalues. Yet, due to the properties of Hermitian matrices, these eigenvalues are always real. To illustrate, consider a generic Hermitian matrix:

$$B = \begin{bmatrix} p & q + ir \\ q - ir & s \end{bmatrix}$$

Where $p, q, s \in \mathbb{R}$ and $r \in \mathbb{R}$ is the imaginary part. Even with complex entries, the eigenvalues of B are guaranteed to be real. Consequently, B is diagonalizable using a unitary transformation.

Examples

Example 1: Consider the complex Hermitian 3×3 matrix M:

$$M = \begin{pmatrix} 0 & 2 + i & 1 \\ 2 - i & 0 & 2 - i \\ 1 & 2 + i & 0 \end{pmatrix}$$

We can diagonalize this matrix with the matrix S and its inverse S^{-1} such that $M = SJS^{-1}$, where J is the diagonal matrix with the eigenvalues of M on its diagonal. The matrix S and its inverse are given by:

$$S = \begin{pmatrix} -1 & 1 & 1 \\ 0 & \frac{(2-i)(\sqrt{41}-3)}{\sqrt{41}-11} & \frac{(2-i)(3+\sqrt{41})}{11+\sqrt{41}} \\ 1 & 1 & 1 \end{pmatrix},$$

$$S^{-1} = \begin{pmatrix} -1/2 & 0 & 1/2 \\ \frac{1}{4} - \frac{1}{4\sqrt{41}} & \frac{-(2+i)}{\sqrt{41}} & \frac{1}{4} - \frac{1}{4\sqrt{41}} \\ \frac{1}{164}(41+\sqrt{41}) & \frac{(2+i)}{\sqrt{41}} & \frac{1}{164}(41+\sqrt{41}) \end{pmatrix}$$

And the diagonal matrix J is:

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2}(1-\sqrt{41}) & 0 \\ 0 & 0 & \frac{1}{2}(1+\sqrt{41}) \end{pmatrix}$$

We have diagonalized M with the matrix S, consistent with the Complex Spectral Theorem.
Example 2: Consider the complex Hermitian 2×2 matrix M':

$$M' = \begin{pmatrix} 1 & 3+i \\ 3-i & 1 \end{pmatrix}$$

We can diagonalize this matrix with the matrix S' and its inverse S'^{-1} such that $M' = S'J'S'^{-1}$, where J' is the diagonal matrix with the eigenvalues of M' on its diagonal. The matrix S' and its inverse are given by:

$$S' = \begin{pmatrix} -\frac{3+i}{\sqrt{10}} & \frac{3+i}{\sqrt{10}} \\ 1 & 1 \end{pmatrix}, \quad S'^{-1} = \begin{pmatrix} -\frac{3/2 - i/2}{\sqrt{10}} & \frac{1}{2} \\ \frac{3/2 - i/2}{\sqrt{10}} & \frac{1}{2} \end{pmatrix}$$

And the diagonal matrix J' is:

$$J' = \begin{pmatrix} 1-\sqrt{10} & 0 \\ 0 & 1+\sqrt{10} \end{pmatrix}$$

We have diagonalized M' with the matrix S', consistent with the Complex Spectral Theorem.

Conclusion

In summary, both the Real and Complex Spectral Theorems guarantee real eigenvalues for their respective classes of matrices. However, the Real Spectral Theorem does so based on the inherent properties of real symmetric matrices, whereas the Complex Spectral Theorem's assurance is intrinsically tied to the Fundamental Theorem of Algebra. This distinction underscores the nuanced interplay between algebraic properties and geometric transformations in the realm of linear algebra.

Applications of The Real Spectral Theorem

The Real Spectral Theorem plays a fundamental role in the diagonalization of real symmetric matrices through real orthogonal transformations. Its broad applicability extends across various fields, including physics, data analysis, engineering, and mathematics. The theorem provides critical insights into the structure and behavior of systems governed by symmetric matrices. In this section, we explore its application in three distinct areas: the second derivative test in multivariable calculus, principal component analysis, and the eigenvalues and eigenvectors of electrical networks.

Second Derivative Test in Multivariable Calculus

The second derivative test in multivariable calculus is an excellent example of the practical use of the Real Spectral Theorem[4]. The purpose of this test is to classify a given critical point of a function as a local minimum, maximum, or saddle point.

A critical part of this process involves the Hessian matrix[5]. This matrix is formed by arranging all second partial derivatives of the function into a square matrix. For a function $f: \mathbb{R}^n \to \mathbb{R}$, the Hessian matrix H at a critical point x is constructed as follows:

$$H(f)(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{bmatrix}$$

The Hessian matrix has the property of symmetry since swapping the order of differentiation does not alter the result, assuming the function is smooth. Consequently, the Real Spectral Theorem can be employed to orthogonally diagonalize the Hessian matrix.

The last step of the second derivative test involves the eigenvalues of the Hessian matrix. These values dictate the classification of the critical point: a local minimum is indicated by all positive eigenvalues, a local maximum by all negative eigenvalues, and a saddle point by a mix of both.

The Real Spectral Theorem ensures that these eigenvalues are real numbers, making the second derivative test a powerful tool for identifying and classifying critical points of a multivariable function.

Principle Component Analysis (PCA)

Principal Component Analysis (PCA) is a foundational technique in data analysis and dimensionality reduction. It aims to identify 'principal components' — directions that capture the most variance in data — facilitating the representation of high-dimensional data in a simplified manner.

To achieve this, PCA computes the eigenvalues and eigenvectors of the data's covariance matrix[6], Σ:

$$\Sigma = \frac{1}{n-1}\mathbf{X}^T\mathbf{X}$$

Where X is the zero-mean data matrix and n is the number of data points. Given the symmetric nature of Σ, the Real Spectral Theorem assures its orthogonal diagonalizability and the reality of its eigenvalues.

In PCA, the principal components are derived from the eigenvectors associated with the largest eigenvalues of Σ[7]. These components enable effective data projection into a lower-dimensional space, maximizing retained variance. Both real eigenvalues and the associated orthogonal eigenvectors are fundamental to this process.

Due to its utility, PCA finds applications in diverse fields such as image processing, finance, and bioinformatics. The Real Spectral Theorem serves as a cornerstone, ensuring the mathematical rigor of PCA.

Eigenvalues and Eigenvectors of Electrical Networks: Kirchhoff's Matrix

In the domain of electrical network theory, the study of networks' topology is often represented using graph theory concepts. The Kirchhoff (or Laplacian) matrix emerges as a central tool, capturing the interplay of connections in the network. Thanks to the symmetric nature of the Kirchhoff matrix, the Real Spectral Theorem finds a pertinent application here.

Given a graph representation of an electrical network, the Kirchhoff matrix, denoted as L, can be defined as:

$$L = D - A$$

where D signifies the degree matrix and A represents the adjacency matrix.

The eigenvalues and eigenvectors of the Kirchhoff matrix offer invaluable insights into the network's properties[8]. For instance:

1. The smallest non-zero eigenvalue is informative of the network's effective resistance.
2. The count of zero eigenvalues reveals the number of connected components in the graph.

Elucidating further, the eigenvectors associated with these eigenvalues can be interpreted as voltage distributions across the network, providing a deeper understanding of network behavior under various conditions. Utilizing the Real Spectral Theorem, one can assure the eigenvalues of L are real, fostering accurate interpretations and applications in the context of electrical networks.

Applications of The Complex Spectral Theorem

The Complex Spectral Theorem, elucidating the diagonalizability of normal matrices via unitary transformations, stands as a keystone in numerous mathematical and physical domains. Beyond its immediate mathematical implications, its prowess shines in quantum mechanics, particularly when navigating the Schrödinger equation. It is worth noting that, in the standard context, the Hamiltonian in the Schrödinger equation is a Hermitian operator. Additionally, the theorem's insights simplify computational challenges, making it instrumental in revealing the dynamics of quantum systems within the rich structure of Hilbert spaces.

The Schrödinger Equation

The Schrödinger equation stands as a foundational component in quantum mechanics, dictating the temporal evolution of quantum systems. Quantum states are described through wave functions, with the Hamiltonian operator, a Hermitian operator, serving as the linchpin in governing their evolution[9].

The Complex Spectral Theorem, which states that every Hermitian operator has a complete set of orthonormal eigenvectors and real eigenvalues, plays a pivotal role in understanding the properties of the Schrödinger equation[10]:

3. Eigenvalue Problem: The time-independent Schrödinger equation is expressed as:

$$\hat{H}\psi = E\psi$$

Here, $\widehat{H}$ denotes the Hamiltonian operator, ψ is the system's wave function, and E corresponds to the energy eigenvalues. The Complex Spectral Theorem is pivotal in guaranteeing that the Hamiltonian's eigenvalues E, representing physically measurable energies, are real. This assurance stems from the theorem's stipulation concerning the real nature of eigenvalues for Hermitian operators.

4. Orthonormal Eigenstates: The eigenstates associated with a specific Hamiltonian, which serve as solutions to the Schrödinger equation, can be made orthonormal. This orthogonality and the complete set of eigenvectors for the Hamiltonian, as vouched for by the Complex Spectral Theorem, play a vital role in quantum mechanics. Specifically, any quantum state can be expansively represented as a linear combination of these eigenstates.
5. Time Evolution: The time-dependent Schrödinger equation delineates the dynamics of quantum state evolution. By expressing a quantum state concerning the eigenstates of the Hamiltonian, future states of the system can be effectively predicted. This process is streamlined by the Complex Spectral Theorem's assurances regarding the properties of the eigenstates and eigenvalues of Hermitian operators.

Briefly, the Complex Spectral Theorem furnishes the Schrödinger equation with a rigorous mathematical framework, certifying that its solutions comport with the tenets of physically realizable and interpretable scenarios.

Conclusion

The spectral theorem is a cornerstone in linear algebra and functional analysis, bridging the theoretical with the practical, and the abstract with the applied. Both its real and complex incarnations, grounded in shared principles, display their unique nuances. The distinctions between them find resonance across diverse fields.

The real spectral theorem, anchored in real symmetric matrices, has applications spanning multivariable calculus, data analysis, and electrical network theory. Conversely, the complex variant, harmoniously aligning with the fundamental theorem of algebra, unveils quantum mechanics' intricacies and fortifies Hilbert spaces' structure. This duality displays the breadth and depth of mathematics, where one theorem can illuminate two vastly different realms.

From exploring the subtleties of eigenvalues and eigenvectors to diving into real-world applications like PCA and the Schrödinger Equation, this paper highlights the intertwined nature of mathematical theory and practice. The spectral theorem exemplifies the unity of mathematical concepts, connecting disparate ideas and shedding light on unexplored avenues. For all who engage with mathematics, it serves as a reminder of the discipline's enduring elegance and depth.

Acknowledgments

I would like to thank my advisor for the valuable insight provided to me on this topic.

References

[1] Sheldon Axler, Ph.D., "Linear Algebra Done Right," Springer, 3rd edition, 2015, https://archive.org/details/SheldonAxlerAuth.LinearAlgebraDoneRight/mode/2up.

[2] Tom Crawford, Ph.D., "Oxford Linear Algebra: Spectral Theorem Proof," YouTube, 2023, https://www.youtube.com/watch?v=ADwsk9G5s_8.

[3] Oliver Knill, Ph.D., *Linear Algebra and Vector Analysis*, Harvard University, 2019, https://people.math.harvard.edu/~knill/teaching/math22b2019/handouts/lecture17.pdf.

[4] Paul Siegel, Ph.D., "Applications of spectral theorem," Mathematics Stack Exchange, 2014, https://math.stackexchange.com/questions/775351/applications-of-spectral-theorem.

[5] *Hessian Supplement*, Harvard University, n.d., math.harvard.edu/archive/21b_spring_02/supplements/hessian.pdf.

[6] David Austin, Ph.D., "7.3: Principal Component Analysis," Mathematics LibreTexts, last updated Sep 17, 2022, https://math.libretexts.org/Bookshelves/Linear_Algebra/Understanding_Linear_Algebra_(Austin)/07%3A_The_Spectral_Theorem_and_singular_value_decompositions/7.03%3A_Principal_Component_Analysis.

[7] Mike Mei, Ph.D., "Principal Component Analysis," University of Chicago, 2009, https://math.uchicago.edu/~may/VIGRE/VIGRE2009/REUPapers/Mei.pdf.

[8] Nikki Carter, "Eigenvalues and Eigenvectors of Electrical Networks," University of Washington, 2015, https://sites.math.washington.edu/~reu/papers/2015/nikki/nikki.pdf.

[9] Jeffrey R. Chasnov, Ph.D., "9.8: The Schrödinger Equation," Mathematics LibreTexts, last updated Dec 5, 2019, https://math.libretexts.org/Bookshelves/Differential_Equations/Differential_Equations_(Chasnov)/09%3A_Partial_Differential_Equations/9.08%3A_The_Schrodinger_Equation.

[10] Seymour Blinder, Ph.D. and David M. Hanson, Ph.D., "3.03: The Schrödinger Equation is an Eigenvalue Problem," Chemistry LibreTexts, last updated Dec 5, 2019, https://chem.libretexts.org/Bookshelves/Physical_and_Theoretical_Chemistry_Textbook_Maps/Physical_Chemistry_(LibreTexts)/03%3A_The_Schrodinger_Equation_and_a_Particle_in_a_Box/3.03%3A_The_Schrodinger_Equation_is_an_Eigenvalue_Problem.