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Simulating Chromatic Harmony in Romantic Era Music using Diophantine Approximation

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ABSTRACT

The Romantic period of music is known for its intricate and emotionally expressive harmonic style. Nevertheless, recreating authentic melodies of the Romantic period in contemporary compositions poses a considerable challenge for composers and musicians. In this paper, we present a method for simulating chromatic harmony in Romantic Era music through the use of Diophantine Approximation. Inspired by the works of renowned composers of that period, such as Richard Wagner, Franz, Liszt, and Frederic Chopin, my transitional model was constructed to preserve the essence of the preceding melody as the harmonic progression unfolds. The model helps with the creation of harmonic progressions that contain the nuances of the preceding harmonic structure and musical style, providing composers with a novel way to explore a wide range of musical possibilities. Experimental validation involving human hearing shows that the model is successful in imitating authentic Romantic-era harmonic structures. This finding suggests that the model is a promising tool that could inspire contemporary composers to create almost authentic chromatic harmonic progressions.

A brief introduction to the Chromatic Harmony in Romantic Era Music

Chromatic harmony in music theory utilizes chords and chord progressions that incorporate notes beyond the diatonic scale of the key employed in a musical composition. In contrast to the conventional practice of diatonic chords, which strictly adhere to the scale's notes, chromatic harmony integrates accidentals or modified notes, augmenting the music with enhanced depth, richness, and emotional fervor. This harmonic technique gained notable prominence during the Romantic Era, a historical period in music known for its emphasis on expressive and emotive compositions.

During the early era of Romanticism, composers like Franz Liszt and Frederic Chopin began utilizing chromatic modulation to transition between distantly related keys, a technique that was later embraced by Richard Wagner. This inventive use of chromaticism in their compositions added an extra dimension to their musical expression, enabling smooth and emotionally charged transitions between remote tonalities. In Wagner's operas, he employed chromatic harmonies to intensify emotional expression and create a sense of dramatic tension.

Figure 1: The opening of Wagner's music drama *Tristan und Isolde*

He adeptly utilized chromaticism to defy traditional harmonic rules (see Figure 1), becoming a pivotal moment in the opera that evokes a profound sense of yearning and desire. Liszt's approach to chromatic harmony surpassed mere technical display. He employed chromaticism to evoke a wide range of emotions and moods, pushing the boundaries of traditional harmonic language. His works often featured daring chromatic modulations and harmonic shifts, creating a sense of unpredictability and intensity. Moreover, Liszt's use of chromatic harmony influenced other composers of his time, including Richard Strauss and Gustav Mahler, who continued to explore and expand the expressive possibilities of chromaticism in their own compositions.

The emergence of chromatic harmony during the Romantic Era gave rise to the creation of chromatic chords, which consist of notes outside the natural scale of the key. These chords frequently involve augmented or diminished intervals, resulting in notably distinct emotional subtleties when compared to traditional diatonic chords. The inclusion of chromatic chords expanded composers' range of colors and emotions, opening doors to more possible territories of expression.

The paper delves into the use of mathematics to model chromatic harmony for multiple reasons. First and foremost, music and mathematics have a long history of interconnectedness, dating back to the ancient Greeks and Pythagorean discoveries. The application of mathematical principles to music provides a systematic approach to understanding and analyzing complex harmonic structures, such as those found in chromatic harmony. By using mathematical models, we can explore the relationships between chromatic chords and their diatonic counterparts, shedding light on the underlying principles guiding the harmonic choices made when incorporating chromatic elements. Thus, some simulated chords will preserve the presence of chromatic harmony due to the existence of chromatic harmonies in the original chords.

Additionally, mathematics allows researchers to quantify and measure the degree of harmonic tension created by chromatic progressions. Moreover, mathematical analyses offer a level of objectivity and precision, allowing composers to identify specific patterns and structures.

In conclusion, the use of chromatic harmony allowed composers to expand beyond the confines of traditional tonal systems, evoking a wide range of feelings and emotions. Utilizing mathematics to model chromatic harmony grants researchers a deeper comprehension of the principles and structures that govern this harmonic language, offering insights into the creativity of Romantic composers.

Fundamentals of Diophantine Approximation

Diophantine approximation is a branch of number theory that deals with the approximation of real numbers using rational numbers. The fundamental concept revolves around finding rational approximations in close proximity to irrational numbers with a good degree of accuracy. In this context, continued fractions play a significant role, providing an efficient approach to approximate irrational numbers.

To illustrate the role of continued fractions in diophantine approximation, let's consider the Golden Ratio, denoted by the symbol ϕ. The Golden Ratio is an irrational number defined as the positive root of the equation $\phi^2 = \phi + 1$, such that it satisfies $\phi \in \mathbb{R} \notin \mathbb{Q}$. It has been widely used by composers and musicians to determine the lengths of different sections within a musical piece. Some musicians have also explored the application of the golden ratio in establishing connections between distinct pitches or intervals within a melody. By following the proportions of the golden ratio, musicians can craft compositions that feel aesthetically pleasing and well-balanced. We can express ϕ as a continued fraction,

$$\phi = 1 + \frac{1}{\frac{2}{\sqrt{5}-1}} = 1 + \frac{1}{1 + \frac{1}{1 + \cdots}}$$

These numerators and denominators of the continued fraction sequence correspond to the Fibonacci numbers. Let the denominators be denoted by Q_i, we have $Q_1 = F_1$, $Q_2 = F_2$, $Q_3 = F_3$, ..., $Q_i = F_i$.
And let the numerators be denoted by P_i, we have $P_1 = F_2$, $P_2 = F_3$, ..., $P_i = F_{i+1}$. As the sequence progresses, the continued fraction yields an infinite sequence of ones,

$$\frac{P_i}{Q_i} = [1;\ 1,1,1,1,\ldots] = 1 + \frac{1}{1 + \frac{1}{1+\cdots}}$$

Notice that the convergents $\frac{P_i}{Q_i}$ get closer and closer to ϕ. In fact, it can be shown that:

$$\lim_{n\to\infty} \frac{P_{n+1}}{Q_{n+1}} = \phi$$

Thus, we can get the difference between the convergents and ϕ approaches zero as n goes to infinity,

$$\lim_{n\to\infty} \left|\frac{P_{n+1}}{Q_{n+1}} - \phi\right| = 0$$

This can be perceived as a form of diophantine approximation because it used the fraction $\frac{P_i}{Q_i} \in \mathbb{Q}$ to approximate the irrational value ϕ. The rational approximations to irrational numbers are represented by convergents of continued fractions. Let $\alpha \in \mathbb{R} \notin \mathbb{Q}$, we make a conjecture that α is between $\frac{P_n}{Q_n}$ and $\frac{P_{n+1}}{Q_{n+1}}$. Thus, we can have,

$$\left|\frac{P_n}{Q_n} - \alpha\right| \le \left|\frac{P_{n+1}}{Q_{n+1}} - \frac{P_n}{Q_n}\right| = \frac{1}{Q_n Q_{n+1}}$$

Since we want to seek out a systematic approach to finding the best possible rational approximations, represented by fractions P_n/Q_n, which come increasingly close to a given irrational number α. We shall see shortly that the inequality above can have only finitely many rational solutions.
Lemma 1 *For any irrational number $\alpha \in \mathbb{R} \notin \mathbb{Q}$, there exists an infinite sequence of rational numbers $\frac{P_i}{Q_i}$, such that,*

$$\left|\frac{P_i}{Q_i} - \alpha\right| \le \frac{1}{(Q_i)^2}$$

This lemma provides a solution to our goal of finding a perfect rational fraction, allowing us to approximate irrational numbers with an ever-increasing level of accuracy. As we delve deeper into this notion of measuring accuracy, we are inspired to seek a more optimized bound for the difference between α and its rational approximations, because the measure of the accuracy of the approximations is of significant interest. Generalize the inequality above, we can introduce the following theorem,

Theorem 2 *For any irrational number $\alpha \in \mathbb{R} \notin \mathbb{Q}$, there exists an infinite sequence of rational numbers p/q for $p,q \in \mathbb{Z}$ and $q \neq 0$, such that,*

$$\left|\frac{p}{q} - \alpha\right| \le \frac{1}{q^2}$$

The expression $\left|\frac{p}{q}-\alpha\right|$ states that the difference between the irrational number α and its rational approximation p/q is less than or equal to $1/q^2$. In other words, the expression $\left|\frac{p}{q}-\alpha\right| \leq \frac{1}{q^2}$ is the tightest bound possible for the approximation error and the rational fraction p/q is a close approximation to the irrational number α, and the difference between them diminishes as the denominator q increases. However, it is important to note that this inequality does not hold true for all irrational numbers α. There are many irrational numbers for which the difference $\left|\frac{p}{q}-\alpha\right|$ doesn't have a tightest bound $1/q^2$.

This leads us to the introduction of the expression $\frac{1}{\sqrt{5}q^2}$. For certain α, a function that provides a tighter upper bound for the difference $\left|\frac{p}{q}-\alpha\right|$. This introduces us to the Hurwitz inequality for Diophantine approximations.

Theorem 3 *For irrational number $\alpha \in \mathbb{R} \notin \mathbb{Q}$, there exists an infinite sequence of rational numbers p/q for p,q $\in \mathbb{Z}$ and $q \neq 0$, such that,*

$$\left|\frac{p}{q}-\alpha\right| \leq \frac{1}{\sqrt{5}q^2}$$

The Hurwitz inequality is a fundamental theorem that provides an upper bound for the difference between an irrational number α and its best rational approximation $\frac{p}{q}$. It plays a crucial role in understanding the distribution of rational approximations to irrational numbers and their degree of accuracy. For example, the Golden Ratio $\phi = \frac{1+\sqrt{5}}{2}$ is one of the irrational numbers for which the diophantine approximation is optimized by the inequality in the above theorem such that the optimized diophantine approximation for ϕ is represented by $\left|\phi - \frac{p}{q}\right| \leq \frac{1}{\sqrt{5}q^2}$.

Overview of Mathematical Parameters of Chromatic Harmony

Chromatic harmony, an aspect of music theory delving into the utilization of non-diatonic chords and. Gaining insight into the mathematical aspects of chromatic harmony equips, musicians and composers obtain valuable knowledge concerning the intricate interplay among chords and progressions. Through a mathematical analysis of these harmonic structures, artists can discover their composing potential and enhance their comprehension of music's emotional and structural elements.

Within the domain of chromatic harmony, chords are formed by incorporating notes from a certain key. By expressing musical notes as numerical entities, like pitch class integers or frequencies, mathematicians and musicians can study the intervals and connections between chords. This method can help musicians discover captivating symmetries, patterns, and repetitions in chord progressions.

One approach is to use a numerical representation for every 12 musical notes within an octave, ranging from 0 to 11. This system allows us to create a discrete set, which can be represented using the mathematical concept of $\mathbb{Z}_{12}$. Each element in this system corresponds to a specific note, and by manipulating these numerical representations, we can analyze the intervals and progressions between chords in a mathematically precise manner.

To establish a connection between musical theory and mathematical structures, various aspects of music, such as MIDI numbers and frequencies, are utilized. MIDI (Musical Instrument Digital Interface) numbers provide a standardized way to represent musical notes across different devices and software. Let S denote the set of MIDI numbers of all notes, we see that the mapping $S \mapsto \mathbb{Z}_{12}$ is surjective, every element $y \in \mathbb{Z}_{12}$ can be mapped from some element $x \in S$ such that $f(x) = y$. And there doesn't exist another representation in $\mathbb{Z}_{12}$ for each element in S, allowing us to explore the inherent symmetries and patterns within chromatic harmony. Since each MIDI note corresponds to a unique pitch, by mapping these MIDI numbers onto the numerical range of 0 to 11, we can seamlessly integrate musical concepts into mathematical frameworks.

Consider a chord progression in the key of C major, incorporating chromatic harmony. We'll start with a C major chord (C-E-G) and then proceed to an unexpected chromatic chord, such as C♯ diminished (C♯-E-G). The numerical representation of these notes can be as follows: C = 0, C♯ = 1, E = 4, and G = 7. By analyzing the intervals between these notes, we can express the chromatic harmony mathematically. For instance, the C major chord has the following interval pattern: (4−0) and (7−4). On the other hand, the C♯ diminished chord features the intervals (4−1), and (7−4). This mathematical analysis reveals how the chromatic chord introduces new approaches to modeling the pattern of music elements, expanding the harmonic palette beyond the traditional diatonic chords.

Thus, for all chords, there exists a representation consisting of numerical integers from 0 to 11 inclusively that can be used to represent the chord. For example, there always exists $a,b,c,d \in \mathbb{Z}_{12}$ such that (E-G-B-E)= $\{a,b,c,d\}$.

The foundation of chromatic harmony lies on harmonic ratios, which govern the relationships between musical frequencies and dictate the consonance or dissonance of intervals. These ratios can be mathematically represented as fractions, with the most harmonically pleasing intervals characterized by the simplest ratios. For example, the perfect fifth, a fundamental interval in music, has a harmonic ratio of 3 : 2, meaning that the frequency of the higher note is $\frac{392.00}{261.63} \approx 1.5$ times that of the lower note. By exploring the properties of harmonic ratios, musicians are able to make great decisions about chord progressions while composing, enhancing the emotional impact and overall coherence of their compositions.

Let's consider the harmonic ratios of a common progression, the major triad. In the key of C major, the C major triad (C-E-G) consists of the following harmonic ratios:

Measuring Harmonic Ratios for different Intervals

- The interval between C and E is a major third, with a harmonic ratio of 5 : 4. This means the frequency of E is $\frac{329.63}{261.63} \approx 1.25$ times that of C.
- The interval between C and G is a perfect fifth, with a harmonic ratio of 3 : 2. This means the frequency of G is $\frac{392.00}{261.63} \approx 1.5$ times that of C.
- The interval between E and G is a minor third, with a harmonic ratio of 6 : 5. This means the frequency of G is $\frac{392.00}{329.63} \approx 1.2$ times that of E.

The consonance and stability of the major triad are governed by these harmonic ratios, resulting in its pleasant and harmonious sound. With this understanding, composers can utilize this information to craft chord progressions that evoke distinct emotions and musical aesthetics.

In practical applications, a more intricate scenario might involve examining the harmonic ratios of extended chords, such as seventh chords or altered chords, whose mathematical representations become more elaborate yet equally insightful for understanding harmony.

The mathematical exploration of chromatic harmony extends to employing various mathematical operations like addition, subtraction, multiplication, and division, strategically crafting chord progressions that smoothly transition between diverse tonal centers. An example of this is the concept of modulations, wherein one key is transformed into another through mathematical operations, effectively shifting the musical focus.

Furthermore, mathematical tools, such as Fourier analysis, can be skillfully utilized to deconstruct intricate harmonic structures into their fundamental frequencies, thus illuminating the spectral content of a chord and its overall harmonic color. This analytical approach uncovers the intricate interplay of musical elements and provides valuable insights into the inner workings of complex musical compositions.

1 Simulating Chromatic Harmony with Diophantine Approximation

To predict the next chord in a chromatic harmony progression, we can employ a mathematical approach that utilizes intervals and ratios between successive chords. The first step is to represent each chord as a set of numbers that contain the representation for each note in the chord. This allows us to capture the unique notes present in each chord, considering their relationships to each other.

Once we have a numerical representation of successive chords, we calculate the intervals between these chords. By determining the pitch differences between consecutive chords, we gain valuable insights into the harmonic movement and development of the music.

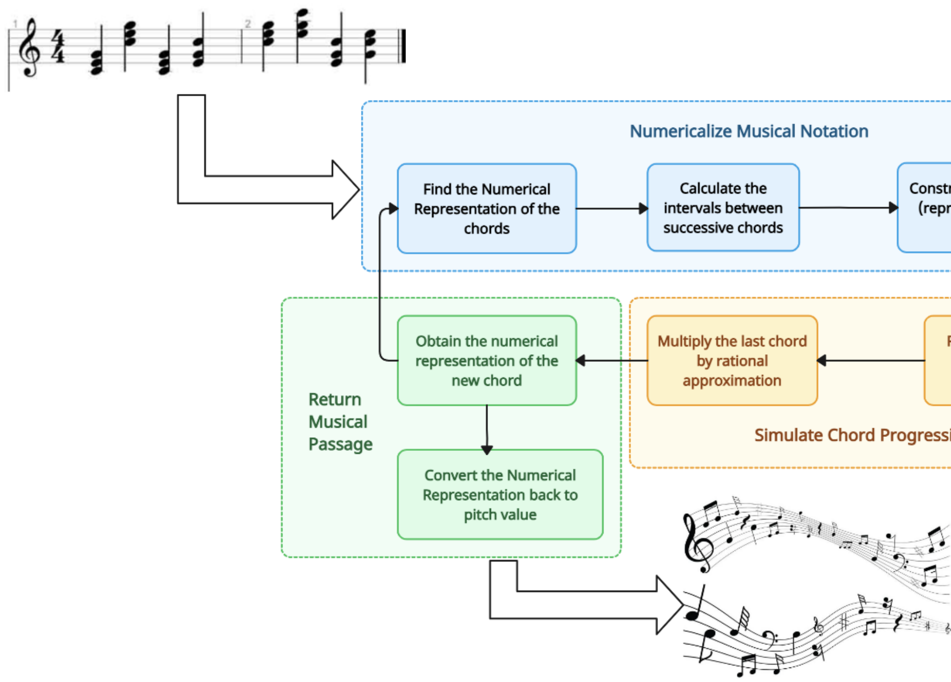

Figure 2: Overview of the process of modeling the chromatic chord progression

As shown in Figure 2, constructing a continued fraction based on these intervals is the next step of the modeling process. The continued fraction represents the ratios between different chords, which could further help our diophantine approximation. The goal is to find p,q $\in \mathbb{Z}$ (where $q \neq 0$) that satisfies the diophantine approximation inequality. This means we have to find the optimized rational approximation that describes the harmonic transition from the current chord to the next one. Applying the ratio p/q to the last chord enables us to predict the pitch values for the next chord in the chromatic harmony progression.

We use the opening chords of Prelude Op. 28, No. 20 in C minor written by Frederic Chopin as an example of modeling because it contains chromatic harmonies. For example, the piece begins in the key of C minor, but the seventh, eighth, and ninth chord contains flat D, which is a chromatic note in the key of C minor. In the C minor scale, the D is natural, but here it is altered to D$\flat$, thus, creating chromatic harmony.

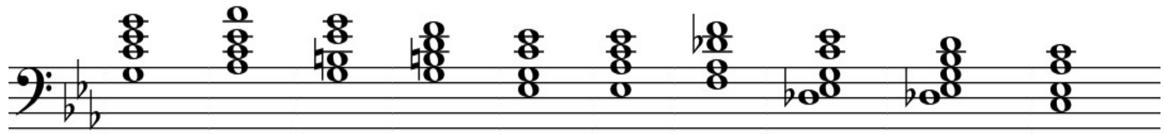

Figure 3: Chords in the opening of Prelude Op. 28, No. 20 in C minor by Frederic Chopin

We write down the notes of each chord in the opening of the Prelude (see Figure 3). The first chord consists of the notes G - C - E♭ - G. Moving forward, the second chord encompasses A♭ - C - E♭ - A♭. The third chord, G - B - E♭ - G, contains the natural B in the harmonic scale of C minor, adding complexity to the melody. We do the same to all the chord, we get the harmonic progression in this form:
(G - C - E♭ - G) → (A♭ - C - E♭ - A♭) → (G - B - E♭ - G) → (G - B - D - F) →
(E♭ - G - C - E♭) → (E♭ - A♭ - C - E♭) → (F - A♭ - D♭ - F) →
(D♭ - E♭ - G - C - E♭) → (D♭ - E♭ - G - B♭ - D♭) → (C - E♭ - A♭ - C)

To facilitate further mathematical calculations and analysis, it is essential to find a numerical representation for each chord in the opening of Chopin's Prelude Op. 28, No. 20. Assigning numerical values to the notes will allow us to transform the musical data into a format that can be processed and manipulated using mathematical operations and algorithms.

One approach to find the MIDI number corresponding to each note in the chord. In this system, each note is assigned a number in $\mathbb{Z}_{12}$, representing its position within the chromatic scale. For instance, the key C4 is assigned the number 60, C♯4 (or D♭) is 61, D4 is 62, and so on. Then we can represent the chords as a set of numbers, allowing us to perform mathematical operations on these numerical representations.

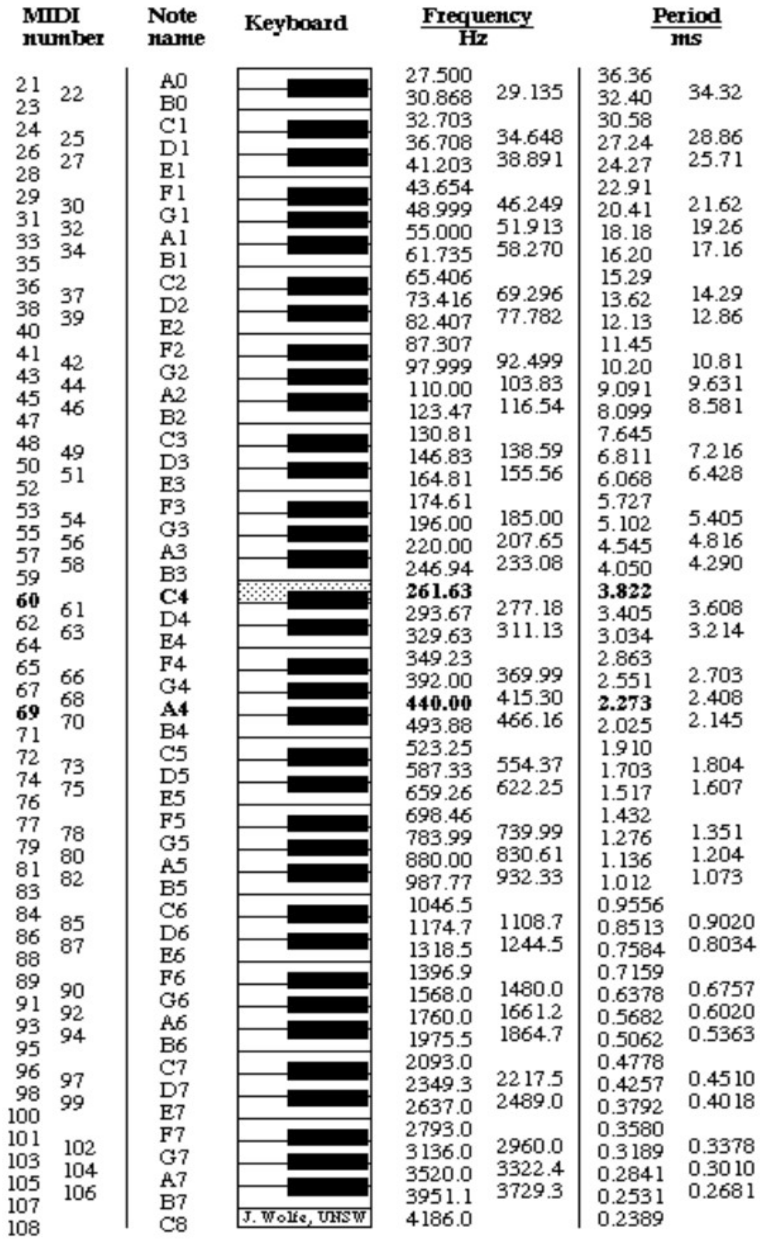

Figure 4: Reference Chart of the MIDI numbers corresponding to the Note name

We take the value of the MIDI number modulo 12, we can then get a number in $\mathbb{Z}_{12}$. For instance, the corresponding MIDI number of A4 is 69 (see Figure 4) and $69 \equiv 9 \pmod{12}$, so we can rewrite the value of A4 as 9. Thus, we can rewrite the chords into different sets consisting of numbers in $\mathbb{Z}_{12}$.

We next construct the continued fraction use the numerical values in each chord. Continued fractions are a special type of fraction where the numerator and denominator are integers, and the fraction is expressed in a specific infinite form.

To begin the process, we take the numerical representation of each chord and create a sequence of integers. Using the reference provided in Figure 4 and the examples mentioned earlier, we get that the numerical representations are as follows:

Chord 1: (G - C - E♭ - G) = {7,0,3,7}
Chord 2: (A♭ - C - E♭ - A♭) = {8,0,3,8}
Chord 3: (G - B - E♭ - G) = {7,11,3,7}
Chord 4: (G - B - D - F) = {7,11,2,5}
Chord 5: (E♭ - G - C - E♭) = {3,7,0,3}
Chord 6: (E♭ - A♭ - C - E♭) = {3,8,0,3}
Chord 7: (F - A♭ - D♭ - F) = {5,8,1,5}
Chord 8: (D♭ - E♭ - G - C - E♭) = {1,3,7,0,3}
Chord 9: (D♭ - E♭ - G - B♭ - D♭) = {1,3,7,10,1}
Chord 10: (C - E♭ - A♭ - C) = {0,3,8,0}

Then we need to determine the musical distance or pitch difference between the numerical representations of the chords. We can then calculate the interval for each pair of successive chords in terms of semitones (half-steps).

- Interval between Chord 1 and Chord 2:
  [8, 0, 3, 8] - [7, 0, 3, 7] = [1, 0, 0, 1]
- Interval between Chord 2 and Chord 3:
  [7, 11, 3, 7] - [8, 0, 3, 8] = [-1, 11, 0, -1] = [11, 11, 0, 11]

Now, we can construct continued fractions for each chord. The general form of a continued fraction is given by $[a_0; a_1, a_2, a_3, \ldots]$, where a_0 is the integer part and $a_1, a_2, a_3, \ldots$ are the continued fraction coefficients. We find the coefficients by iteratively dividing the integer part of the number by the fractional part.

We find each value of a_n with $n \geq 0$ from each interval between successive chords. Suppose the sequence of integers $[x_1, x_2, x_3, x_4]$ for $x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}$ is the interval between nth chord and chord $(n+1)$th chord, we have $a_n \in \{x_1, x_2, x_3, x_4\}$. Similarly, given that the sequence of integers $[y_1, y_2, y_3, y_4]$ for $y_1, y_2, y_3, y_4 \in \mathbb{Z}_{12}$ is the interval between $(n+1)$th chord and chord $(n+2)$th chord, we then have $a_{n+1} \in \{y_1, y_2, y_3, y_4\}$.
Thus, using the sequence of integer $[a_0; a_1, a_2, a_3, a_4, a_5, \ldots]$ we get, we can construct the continued fraction for the change in pitch between chords,

$$\delta = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cdots}}}}}$$

After obtaining an irrational number $\delta \in \mathbb{R} \notin \mathbb{Q}$, we are able to construct the inequality that quantifies how well this irrational number can be approximated by rational numbers with specific properties.
Diophantine approximation focuses on finding rational numbers that come close to irrational numbers. For a given irrational number δ, we aim to find rational numbers p/q (where $p,q \in \mathbb{Z}$) such that the difference between δ and p/q is as small as possible.
One common way to construct the Diophantine approximation inequality is by using the following expression:

$$\left|\delta - \frac{p}{q}\right| \leq \frac{1}{q^2}$$

where δ is the irrational number from the continued fraction, and p/q represents the rational approximation. The inequality states that the difference between the irrational value δ and the rational approximation p/q must be less than the reciprocal of the square of the denominator (q) of the rational approximation.

The Diophantine approximation inequality helps us quantify how closely we can approximate an irrational number using rational numbers. The smaller the value of $1/(q^2)$, the better the approximation. In other words, if we can find rational numbers p/q that satisfy the inequality for increasingly larger values of q, the irrational number δ is said to have "good" Diophantine approximations. And p/q is considered a rational best approximation of δ if,

$$\left|\delta - \frac{p}{q}\right| \leq \left|\delta - \frac{p'}{q'}\right|$$

for all $p', q' \in \mathbb{Z}$, $1 \leq q' \leq q$.

After finding the optimized p/q value we want for δ, we multiply the last chord by the ratio p/q to simulate the harmonic progression of the next chord. This ratio represents the harmonic relationship between the last chord and its harmonic progression, which we can interpret as a "scaling factor" for the chord.

Given the numerical representation of the last chord as $[z_1, z_2, z_3, z_4]$ for $z_1, z_2, z_3, z_4 \in \mathbb{Z}_{12}$, by multiplying the note of the last chord by the ratio p/q, we effectively stretch or compress the chord, altering its pitch and potentially creating new harmonies. This process is akin to a musical transformation, where the irrational value obtained from Diophantine approximation guides the modification of the chord's tonal properties.

The multiplication process will yield a new chord with pitch values $[r_1, r_2, r_3, r_4]$ for $r_1, r_2, r_3, r_4 \in \mathbb{Z}_{12}$. Since $p/q \in \mathbb{Q}$ is a rational approximation of the irrational value associated with the last chord, these new pitch values will be rational numbers, maintaining the original chord's harmonic framework.

Therefore, we can add the simulated chord to the end of the harmonic progression. This new chord, created through the mathematical transformation, becomes a novel addition to the musical sequence. Thus, the resulting harmonic progression now includes both the original chords, followed by the simulated chord derived from Diophantine approximation.

To develop the next chord after the simulated chord, we repeat the exact same steps of Diophantine approximation used previously. We take the numerical representation of some of the previous chord and simulated chord, and seek the best rational approximation p/q to represent its harmonic development. This new ratio p/q serves once again as a "scaling factor" that we can use to multiply the notes of the simulated chord, just like we did with the last chord.

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Inputted Chord Progression:
{7, 0, 3, 7}, {8, 0, 3, 8}, {7, 11, 3, 7}, {7, 11, 2, 5}, {3, 7, 0, 3}, {3, 8, 0, 3},
 {5, 8, 1, 5}, {1, 3, 7, 0, 3}, {1, 3, 7, 10, 1}, {0, 3, 8, 0}

New Next Chord: {1, 3, 10, 1}

New Next Chord: {1, 7, 10, 1}

New Next Chord: {3, 7, 10, 3}

New Next Chord: {7, 10, 2, 7}

New Next Chord: {5, 8, 11, 5}

New Next Chord: {3, 7, 11, 3}
>
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Figure 5: Numerical Representation of Simulated Chromatic Harmony Progression based on the opening of Prelude Op. 28, No. 20, in C minor by Frederic Chopin

Apply this process of Diophantine approximation and chord simulation to the chords in the opening of Chopin's Prelude Op. 28, No. 20 in C minor, we can get a numerical sequences of the next six chords (see Figure 5).

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Noticing that some of the chords were simulated to be chromatic harmonies due to the presence of chromatic harmonies in the original chords.

Then we can use the MIDI number again to convert the numerical representation of the new chords we obtained into musical notes on staff as shown below in Figure 6. The MIDI number system assigns specific integer values to musical pitches, allowing us to accurately transform the numerical chord representations into tangible musical symbols. For instance, suppose the MIDI number of the pitch of the note is a for $a \in \mathbb{Z}_{12}$, we can rewrite a as $a \equiv a + 12k$ (mod 12) for $k \in \mathbb{Z}$. Based on the pitch of the previous chords, we determine the value of k and obtain final pitch of the chord from its numerical representation. The conversion process from numerical data to musical notation provides a visual representation of the harmonic progression, helping us understand the melodic relationships between the chords. It helps provide us with insights into the musical flow and phrasing, highlighting the patterns present in the compositions.

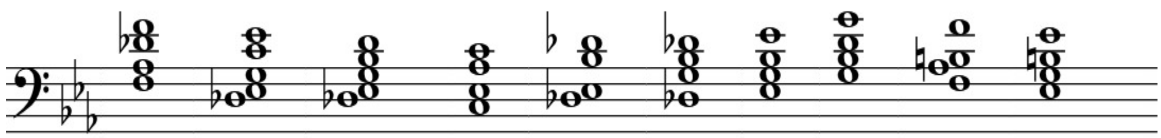

Figure 6: Simulated Chromatic Harmony Progression based on the opening of Prelude Op. 28, No. 20 on staff

Conclusion

In conclusion, the application of Diophantine approximation to simulate chromatic harmony in music has provided us with valuable insights and creative possibilities. By converting musical chords into numerical representations and employing Diophantine approximation to find rational approximations for the change in pitches of the chords, we have successfully simulated new chords that preserve the essence of the original composition including its chromatic elements.

Using this mathematical method, we have unveiled a series of following chords that mirror the harmonic complexities embedded in Chopin's Prelude Op. 28, No. 20 in C minor. The iterative simulation of chords has enabled us to explore the concealed patterns and interconnections within the composition. Notably, certain simulated chords have showcased chromatic harmonies, enriching the musical progression with deep emotional expression. The conversion of the numerical chord representations back into musical notes on a staff using the MIDI number system further enhanced our understanding and appreciation of the harmonic sequence. While the staff notation allowed us to visualize the melodic and harmonic structure, leading to a deeper comprehension of the musical flow and phrasing. The value of employing Diophantine approximation to simulate chromatic harmony lies in the intersection of mathematics and music. This interdisciplinary methodology emphasizes the influence of mathematical concepts in enhancing music analysis and composition. Through mathematical precision, we have discovered novel harmonic connections and melodic motifs within the piece. This newfound comprehension can assist music theorists and composers in delving into the foundational structures of intricate musical compositions.

Moreover, the potential of this method extends beyond analyzing existing music; it presents a distinctive opportunity for composers to craft imaginative and evocative musical passages. The process of simulating chromatic harmony using Diophantine approximation unlocks creative pathways, empowering composers to explore novel harmonic progressions and create original compositions that resonate with audiences. The ability to simulate chromatic harmonies through Diophantine approximation opens up possibilities for composers to fill the gap between theoretical analysis and artistic expression. It encourages a symbiotic relationship between the rich heritage of classical music theory and the ever-evolving contemporary composition.

With the continuous advancement of technology and the fusion of mathematical methods, computer software has the potential to transform music composition and analysis. Computational tools can support composers in delving into a plethora of harmonic possibilities, offering instant feedback and opening the door to the exploration of undiscovered musical realms. Besides that, it also opens up the realm of algorithmic composition. By automating parts of the creative process, composers gain the ability to efficiently delve into a wide range of harmonic possibilities and produce a rich variety of musical material. This approach merges the intuitive and expressive nature of music with the accuracy and efficiency of mathematical algorithms.

In summary, the utilization of Diophantine approximation to simulate chromatic harmony has yielded fruitful results. The interplay of mathematics and music in this pursuit highlights the limitless potential for creative expression and the endless possibilities of composition. This integration serves as a source of inspiration to delve deeper into the connection between art and mathematics, fostering novel pathways of exploration in the realm of music composition.

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