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Paradoxes and their Findings in Special Theory of Relativity

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ABSTRACT

After the publication of Special Theory of Relativity in 1905, scientists raised questions that seemed to be impossible to explain due to relativity. These paradoxes seem to make contradictory statements and challenge the legitimacy of the theory of relativity until it was accepted by the scientific community. This `paper presents some of those paradoxes and findings of Special Theory of Relativity, namely the Bug Rivet Paradox, Bell's Spaceship Paradox and the changes in the electromagnetic fields due to a change in frames of reference. The paper shows how these problems were solved using coordinate transformations, and the key insights into the world we live in, such as the explanations of why electric and magnetic fields are interlinked. Physicists came up with apparent paradoxes which challenged the theory. Apparent paradoxes arise due to two statements that seem logical but do not agree with each other, or an antinomy. We will be looking at some of these problems; Bug-Rivet paradox, Bell's Spaceship paradox and the link between electric and magnetic fields.

Introduction

The first time Special Theory of Relativity was introduced into the scientific community in Einstein's paper with the translated title "On the Electrodynamics of Moving Bodies" published in 1905 from German. In the paper, he proposed two axioms: ther e is no absolute rest, and the speed of light, c, is constant for all frames (Einstein, 1905). In the paper, equations which are called Lorentz transformation can be found, although with a different notation. Special theory of Relativity lays the foundation of topics such as General Theory of Relativity, which includes the effects due to gravity, and other fields in physics.

Physicists came up with apparent paradoxes which challenged the theory. Apparent paradoxes arise due to two statements that seem logical but does not agree with each other, or an antinomy. We will be looking at some of these problems; Bug-Rivet paradox, Bell's Spaceship paradox and the link between electric and magnetic fields.

Background Review

Frames of reference

As one of the axioms states that there is no absolute rest. Thus, statements, such as an object moving at 5km per hour are not viable. This is because such statements need something to be compared against, and in most daily uses, it is the earth, which is presumed to be stationary for most purposes. Similarly in relativity, something that we can compare needs to be created, so that instead of an object moving at 5km per hour, the statement is the object is moving at 5km per hour in respect to the earth.

One problem with this is the number of objects that exist. Multiple objects can exist with no velocity compared to each other. To generalize, and as the effects of special relativity is the same with the same velocity, this can be generalized and called as one frame of reference. Thus, it can be said that the stationary objects are stationary in their frame of reference.

This normally has a symbol, such as frame S, and S' to illustrate variables in each frame of reference. An object with distance x, or time t in frame S, is only valid for one frame, so the variables in a different frame, such as S' has distance x' and time t'.

Spacetime diagram

A spacetime diagram shows the position and time of entities and events in one frame of reference, with position on the x axis and time on the y axis. For example, an object moving at the same speed (in the x axis) as the frame of reference will be straight on the diagram, as it is not moving in respect to each other.

A trajectory is the path an object takes in a frame of reference. This can be shown with a Spacetime diagram, where a line represents the trajectory and a point on the line shows where it is in that time, in their frame of reference.

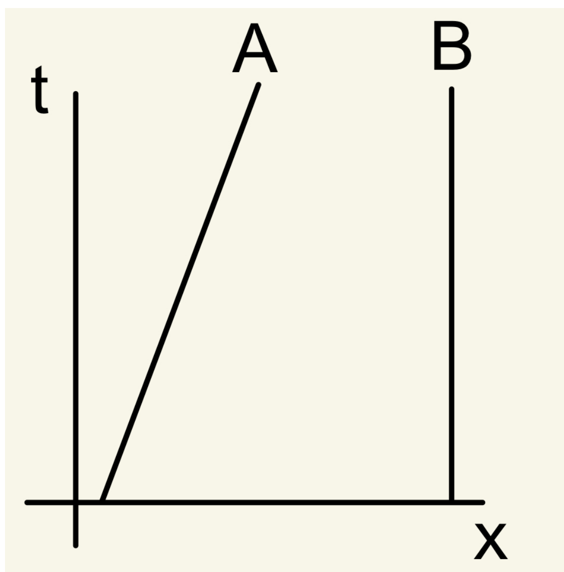

Figure 1. Spacetime diagram. In the figure above, the trajectory of object B is stationary in respect to the frame of reference in which this diagram is drawn. Object A is moving at a constant velocity in the positive direction.

Coordinate transformation

A coordinate transformation refers to (a set of) equations that can give a coordinate change the coordinate from one system to another. For example, if there are two sets of coordinates, one in miles and one in kilometres, and with a corresponding coordinate transformation, the position in one coordinate in terms of miles can be found in terms of kilometres.

One such coordinate transformation is the Lorentz transformation, which is vital in relativity. With a given coordinate in one frame, and other information such as the relative velocity, it is possible to find the coordinate for the other observer, even with the length and time contraction being accounted for.

With v being the velocity of the other object, and X and t being distance and time in their frame of reference,

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

The Lorentz transformation is:

$$X' = \gamma(x - vt)$$

$$t' = \gamma(t - \frac{vs}{c^2})$$

(Jean-Michel, 2007)

Maxwell's equations

Maxwell's equations describe all interactions of electric and magnetic fields in classical mechanics. There is a function, divergence, in the Maxwell's equations symbolised by ∇. This shows the gain/loss of a quantity in an area. For example, a pool with net water change has a divergence of 0, even if the water is flowing, while a pool that has water flowing into it will have a positive divergence. The Maxwell's equations are:

$$\nabla \cdot \vec{B} = 0$$
$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \times \vec{E} = -\frac{\partial B}{\partial t}$$
$$\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

where ρ is the electric charge density and $\vec{J}$ is the current density (Institute of Physics, n.d.).

Bug Rivet paradox

The Setup

There is a T shaped block and a U-shaped block that fits. However, the ends (p and h) do not meet. There is a bug that is on the face of p and at least one of the blocks are travelling towards the other, so the depth of p decreases.

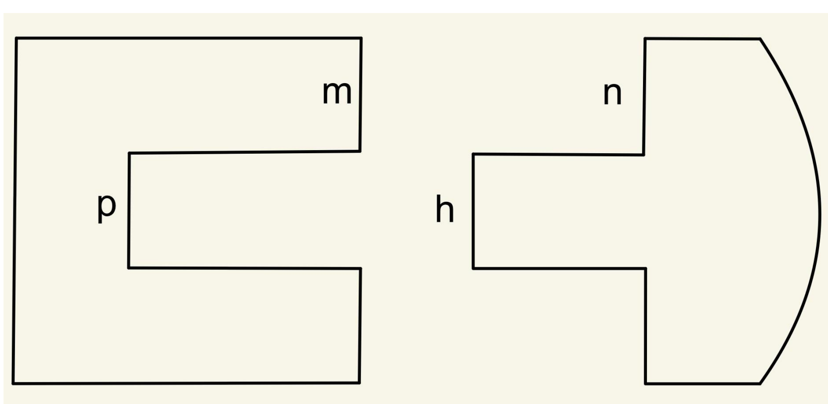

Figure 2. Illustration of the Bug Rivet paradox. The objects are in rest, Figure 3, with the U-shaped block on the left and the T shaped block on the right at rest.

Paradox

In the frame of reference for the T shaped block, S, the depth is reduced due to length contraction, and so p and h meet. However, in the frame of reference for the U-shaped block, S', p and h do not meet as there is no length contraction on the longer side. In addition, the T shaped block is smaller in length, making the fact that it touched more unbelievable.

Qualitative resolution

The paradox comes from the assumption that solids act as a whole instantly when a force is applied in the frame S'. In this case, the assumption is that when a force occurs at m and n, h would also stop moving instantly for all frames of reference. This is not the case, as in frame S', there must be a transmission of information that "tells" atoms at h to experience a force and stop moving. Thus, for S', there is a time period when h is moving after it m and n has collided. This is possible as simultaneity is lost due to relativistic effects (Norton, 2022). The assumption for the frame S made in the paradox is correct, and thus both frames agree that p and h met, although with different reasoning.

Quantitative resolution

Let the length of the U-shaped block be L_R, and the length of the T shape be L, so that $L_R = \gamma L$. This is due to the trajectory of the end points being the same, except for their starting point, and their length in-between is reduced by length contraction. This can also be proved with the Lorentz transformation. In the frame of S, the minimum speed needed, or the critical velocity, is found by the Lorentz transformation:

$$L_R = \gamma L$$

$$L_R/L = \sqrt{1 - v^2/c^2}^{-1}$$

$$(L_R/L)^2 = (1 - v^2/c^2)^{-1}$$

$$(v/c)^2 = 1 - (L_R/L)^2$$

$$v = c\sqrt{1 - (L_R/L)^2}$$

In the frame of S', the time taken for h to reach p must be lower or equal to the time light needs to go from m to n. Thus, the following must be true:

$$\frac{L_R - L/\gamma}{v} \leq L_R/c$$

$$c(L_R - L/\gamma) \leq L_R v$$

$$c(L_R - L/\gamma) \leq L_R \cdot c\sqrt{1 - (L_R/L)^2}$$

$$1 - \frac{1}{\gamma^2} \leq \sqrt{1 - \frac{1}{\gamma^2}}$$

$$(1 - \frac{1}{\gamma^2})^2 - (1 - \frac{1}{\gamma^2})^2 \leq 0$$

Let $x = 1 - \frac{1}{\gamma^2}$ and $\gamma > 1$ when there is motion, so $0 < x < 1$

$$x^2 - x \leq 0$$

$$x(x - 1) \leq 0$$

And this is always true, given the given range. Therefore, the ends of the blocks always meet before light, or any other form of information can be transmitted.

Bell's Spaceship paradox

The Setup

There are two spaceships at rest, facing the same direction. There is a brittle thread hanging in between, so it shatters with the slightest force within the object. In the inertial frame, S, the spaceships accelerate at the same time at the same rate, by a signal from the half of the distance between them. Let the frame of reference for the accelerating spaceships be S'. Frame S and S' are the same until the spaceships start to accelerate.

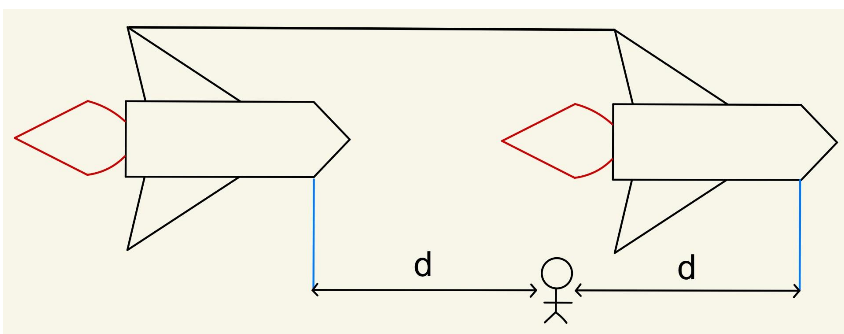

Figure 3. Illustration of Bell's Spaceship paradox. The rod is the connection above, and the blue lines are for illustrative purposes to show the distance between the person, who represents the origin of the signal for both rockets to start accelerating, is the same.

Paradox

In the frame S', the front spaceship accelerates at the same time as the other by the proper acceleration. Their velocities will always be the same, or the other would be a vertical line in one's Spacetime diagram. Thus, their distance in between is constant, as it is the real length, and there is no length contraction. Thus, the thread should not shatter.

However, in the frame S, the distance between them is kept constant, as they are moving the same distance over the same period. This can be illustrated with a Spacetime diagram, where both rockets have the same trajectory, but just start at different distances, resulting in the same gap between them. However, the length of the thread will shrink, due to length contraction, resulting in a stretch and causing a force on the rod, breaking it.

Moreover, for any other frame which is moving in respect to the inertial frame, the spaceships accelerate at different times, due to the loss of simultaneity (Norton, 2022). This leads to two contradictory conclusions, where one frame concludes the thread is not going to shatter, and infinite other frames conclude that it is.

Resolution

The paradox arises from the frame of the spaceships believing the thread will not shatter, and every other frame thinking it will otherwise. However, the assumptions made by the frame of the spaceships are incomplete. When the spaceships accelerate, the rod cannot be thought of as one single entity, but multiple, spread across the thread like the molecules they are made up of.

When the spaceships accelerate, there is a period of time where the ends of the thread either experience push or a pull by the spaceships, but the middle part does not, as there has not been enough time to transmit the information. Though it may be a small window, a part experiences a push without a pull, or vice versa, and thus it creates a force within the thread itself, shattering it due to its brittleness.

Change in the Electromagnetic fields due to Lorentz Transformation

The Setup

At first, electric and magnetic fields seem unrelated. Although they are linked through equations, such as the Maxwell's equations, it may seem similar to a change in the electric field due to a charged object's motion being described by different velocities, or a few equations that show no fundamental link between the fields. Especially with electric fields being created by charged objects, and magnetic fields being created by magnetic objects, most notably by ferromagnetic objects which seem completely unrelated, they seem to have no agreeing source. However, through relativity, one field can appear solely due to movement and the existence of the other field.

Let there be a positively uniformly charged rod that is infinite in length with no ends, parallel to the Z axis. For the stationary observer with respect to the rod, in the frame S, only experiences an electric field. Let S' be a frame in which it moves only in the Z axis in respect to S. To find the exact electric and magnetic field strengths for S and S', which moves at βc in respect to S, the following equations are going to be used.

The electric field, E, and magnetic field, B is calculated from the four potential by the following equations.

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

(Grant & Phillips, 1990). ∇ is the Nabla or Del operator, with divergence and curl being used for last and second last equations respectively.

Where A^{μ} represents the four potential, or the four vector for frame S and where,

$$A^{\mu} = (A^0, A^1, A^2, A^3) = (A^t, A^x, A^y, A^z) = (\phi, \vec{A})$$

$$\vec{A} = (A^x, A^y, A^z) = (0,0,0)$$

and ϕ is

$$\phi = k\, ln(r)$$

(Dray & Monogue, n.d.)

Where $r = \sqrt{x^2 + y^2}$. Thus, the following can be said. This proof will be used multiple times.

$$\frac{\partial ln(r)}{\partial x} = \frac{1}{r}\frac{\partial\sqrt{x^2+y^2}}{\partial x}$$

$$u = x^2 + y^2$$

$$\frac{1}{r}\frac{\partial\sqrt{x^2+y^2}}{\partial x} = \frac{1}{r}\frac{\partial\sqrt{u}}{\partial x}$$

$$\frac{\partial u}{\partial x} = 2x$$

$$\partial x = \frac{\partial u}{2x}$$

$$\frac{1}{r}\frac{\partial\sqrt{x^2+y^2}}{\partial x} = \frac{1}{r}\frac{\partial 2x\sqrt{u}}{\partial u} = \frac{1}{r}\frac{x}{\sqrt{u}} = \frac{x}{r^2}$$

Stationary frame

As this frame is stationary in reference to the rod, as it is in frame S, there is no velocity relative to the rod. Thus, there is no need to transform the values. In the frame S, using the transformation from four potential to the fields, $\vec{E}$ and $\vec{B}$ are:

$$\vec{E} = -(k\frac{\partial ln(r)}{\partial x}, k\frac{\partial ln(r)}{\partial y}, 0) - \frac{1}{c}\cdot 0$$

$$= -(\frac{kx}{r^2}, \frac{ky}{r^2}, 0) = (\frac{kCos\theta}{r}, \frac{kSin\theta}{r}, 0)$$

$$\vec{B} = \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Moving frame

This frame, S', is moving in respect to frame S in the direction of the Z axis, by βc. Thus, there is a need to transform the values into this frame using the Lorentz transformation matrix or transform the values one by one.

$$A'^{\mu} = (\gamma k ln(r), 0, 0, -(\gamma\beta k ln(r))$$

$$\phi' = \gamma k ln(r)$$

To find the electric and magnetic fields, the same four vector transformations can be applied.

$$\vec{E} = -(\frac{\partial \gamma k ln(r)}{\partial x}, \frac{\partial \gamma k ln(r)}{\partial y}, 0) - \frac{1}{c} \cdot (0,0, \frac{\partial \beta \gamma k ln(r)}{\partial t})$$

$$= -(\frac{\gamma k x}{r^2}, \frac{\gamma k y}{r^2}, 0)$$

$$= (\frac{\gamma k Cos(\theta)}{r}, \frac{\gamma k Sin(\theta)}{r}, 0)$$

As S' is only moving in the Z direction in respect to S, $x = x'$ and $y = y'$, and so the same proof was used again. The magnetic field is:

$$\vec{B} = \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & A'^{\mu} \end{pmatrix}$$

$$= \hat{\imath} \frac{\partial A'^{\mu}}{\partial y'} - \hat{\jmath} \frac{\partial A'^{\mu}}{\partial x'}$$

$$= -\hat{\imath} \frac{\partial \beta \gamma k ln(r) + vt}{\partial y'} + \hat{\jmath} \frac{\partial \beta \gamma k ln(r) + vt}{\partial x'}$$

$$= -\hat{\imath} \beta \gamma k \frac{\partial ln(r)}{\partial y} + \hat{\jmath} \beta \gamma k \frac{\partial k ln(r)}{\partial x}$$

$$= -\hat{\imath} \beta \gamma k \frac{Sin(\theta)}{r} + \hat{\jmath} \beta \gamma k \frac{Cos(\theta)}{r}$$

Thus, due to the movement in the Z axis in respect to S, S' will experience a magnetic force specifically due to the movement in the Z direction.

Unification of Electric and Magnetic fields

The two equations showed that the two fields are inseparable; even if one field is 0 in one frame, it may not be in a different frame. With no source of a magnetic field, it can be created by an electric charge moving in respect to an observer. Although relativity was not the first theory or equations to describe electric and magnetic fields with many before it, most notably the Maxwell's equations, it played a significant role. Relativity, which the Maxwell's equations were compatible with, gave an explanation and proved that they are inseparable; magnetic fields are one frame of reference's explanation for another frame of reference's electric field due to length contraction, or the lack thereof.

This can be shown by another thought experiment. Let there be a neutral wire of infinite length which has a current. Let there be a frame S, which is stationary in respect to the wire, and an electron in S', which moves at the same velocity as the electrons in the wire. There is no electric field in the frame of S due to the wire being neutral. However, there is a magnetic field due to Ampere's law. In frame S', due to length contraction of the protons, the wire seems to be charged, as there is a higher density of protons, and the particle experiences a force due to the electric field. (There is also a magnetic field, although it is less). However, in the frame of S, which is stationary in respect to the wire, does not experience the force from the electric field. In frame S, the force on particle in S' cannot be explained just by electric fields. Relativity showed that the force is explained by the magnetic field, and the electric field for one frame of reference is an explanation for a magnetic field in a different frame of reference (Weissman, 2007).

Summery

When we start to think about relativity without changing the axioms from the classical physics that we grew up with, false presumptions are made. Axioms that are found at the start of relativity, such as time being relative, and intuitive, but wrong assumptions, such as agreed instantaneous events in all frames must be re-conceptualized if correct answers are wanted. In this paper, paradoxes were raised due to the wrong assumptions carried over from daily life, which all highlight one important point: The loss of instantaneous events means there are different explanations for one event that has occurred, which is evident in the example of electromagnetism. Along with other assumptions that is not thought of in daily life, it makes paradoxes hard to solve with a mindset with classical mechanics.

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