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General Form Equation for the Most Energy-Efficient Basketball Shot

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ABSTRACT

Although ball shooting is one of the fundamental skills of basketball, it is energy intensive and exhausts players quickly. Because of this, effective shooters will have less time to play on the court, which can be the difference between winning and losing for basketball teams. In the National Basketball League (NBA), pride, expectations, and millions of dollars in investments are at stake, making winning all the more critical. Improving basketball shooting accuracy may provide the most robust path forward to increasing team-specific wins. So, to boost shooter performance, we derived an equation that calculates the optimal velocity to shoot a basketball from every position on the court. This 'optimal' velocity minimizes a shooter's energy expenditure so players can conserve their energy and stay effective in the game longer, boosting their team's chances of winning. We used Newton's second law and kinematics, vector algebra, and calculus to derive the optimal velocity equation and ultimately implemented it in Python for public use. We concluded that the optimal angle for mid-range shots (3 to 5 meters) is 55 to 51 degrees and 49 to 47 for three-point shots.

Introduction

The basketball market is highly lucrative. The average National Basketball League (NBA) team is worth $2.86 billion (Forbes, n.d.). The NBA's revenue in the 2021/2022 season alone topped $10 billion (Ozanian & Teitelbaum). With more wins possibly instigating greater profit, research development in basketball performance optimization has become prevalent.

Over the past decades, substantial technological advances have emerged in the basketball industry to improve the performance of NBA stars and accurately quantify a player's value; there are basketball shoe designs to boost explosive movement and protect against injuries, kinesiological studies on the mechanics of dribbling, passing, shooting, and new models for evaluating the different advantages and disadvantages that each player bears. Many basketball concepts have been distilled to a science, but shooting and the accuracy of a shooting form have remained relatively untouched.

Basketball research has already been conducted for the past several decades. For example, Brancazio (1982) concluded that an understanding of the physics in basketball could improve a player's in-game performance, an angle of 45 to 55° to the horizontal is the optimal energy release angle for 3-pointers, and that the greater the release angle, the more leeway for error. Baca and Perl (2020) investigated performance modeling and simulating a basketball game given specific parameters; McKay (2001) examined the risk factors leading to ankle injuries in recreational basketball players; Li et al. (2022) found that dribbling compromised running speed; Chakraborty and Mondal (2019) analyzed the factors that led to a successful layup; Fujii et al. (2010) observed a sharp decrease in movement and dribbling frequency when players are fatigued; and Altavilla et al. (2017) developed six dribbling drills to practice several fundamental techniques of dribbling.

Despite such rigorous research, the most revolutionary technological developments in basketball are the artificial intelligence-related research projects conducted in the past few years. For example, Raj-Prasad et al. (2016) analyzed Stephen Curry's three-point shots and optimized them, decreasing the energy expenditure by 23% while keeping the fundamental attributes of Curry's shooting form. Cheng et al. (2005) trained an AI computer vision system and control system to shoot a basketball with 90% accuracy. Furthermore, Xu and Tang (2021) developed a path-finding artificial intelligence robot to coordinate the movement of players on the court to score while avoiding collisions and obstacles.

Because of basketball's aggressive nature, basketball shooting has increasingly become a prioritized skill. Over the past few decades, basketball teams are increasingly taking more and more three-point shots (see Figure 1) because of how efficient it is in scoring points.

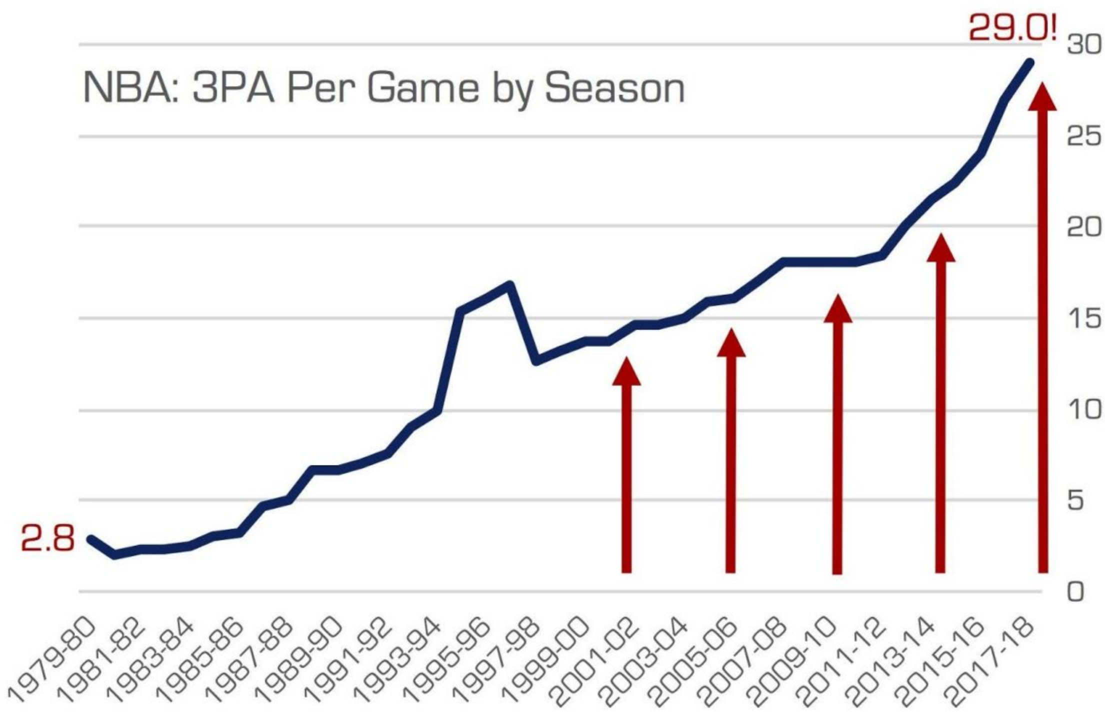

Figure 1. Starting from when the three-point shot was introduced in 1979, the number of three-point attempts per game has multiplied tenfold. Overall, this trend is close to being linear. The spike in three-point attempts from 1994 to 1997 occurred because during those seasons, the distance of the three-point line from the basket was shortened from 23 feet and 9 inches to 22 feet. Shortening the distance made it easier to score three-point shots, making it an even more appealing scoring method. (Figure adapted from The 3 Point Revolution In *ShotTracker* by S. Shea. Retrieved 2019, from https://shottracker.com/articles/the-3-point-revolution. Copyright 2022 by ShotTracker. Reprinted with permission.)

Players are shown to average 1.11 points per 3-point shot and only 0.79 points per 2-point shot (Shea, n.d.). This evaluative metric explains why teams are taking more threes: to score more points expeditiously. Basketball shooting is an energetically expensive activity, with the National Academy of Sports Medicine (n.d.) estimating that shooting for an hour burns about 300 calories. This evidence suggests that basketball shots players would merit from optimizing their energy shot consumption for all shots. Thus, to make 3-point shooting an even more appealing shot selection, we found the optimal velocity (minimum velocity to score a swish) when shooting a basketball for every position on the court.

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Methods

What is a successful shot?

A successful shot must satisfy the following two criteria:

1. The basketball must enter the basket from above and pass through the net.
2. The basketball does not touch the basketball rim at any point in its trajectory.

We ignored shots that are behind the net because such a shot rarely occurs in a basketball game. We solved for and modeled the launch velocity of the basketball using kinematic equations, ignoring air resistance. Furthermore, we did not account for the spin of the basketball to calculate the minimum velocity for a swish because we assumed that their effects are negligible to attempt to achieve the most parsimonious model.

In this study, we grouped shots by the horizontal distance from the basketball to the net, since we can reasonably assume that the shooter always faces toward the net, thereby reducing the three-dimensional physics problem to a two-dimensional one.

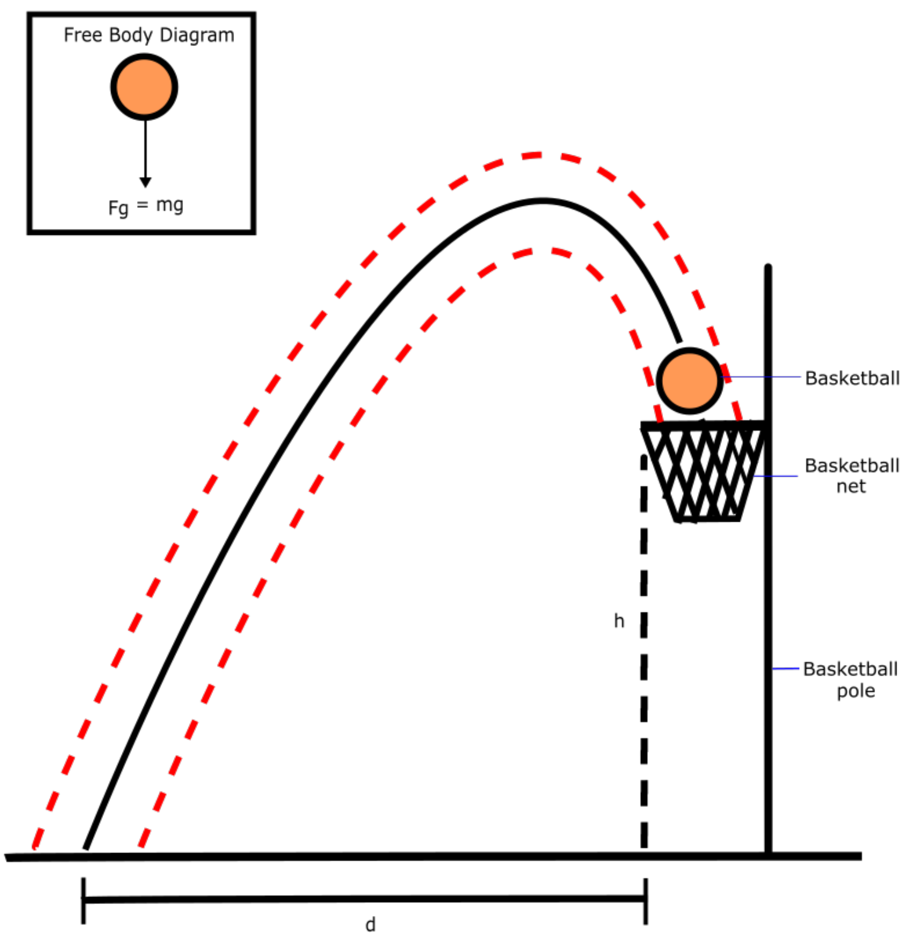

Figure 2. A typical successful shot. In this Figure, d is the horizontal distance from the player's release shot and the front of the rim, h is the height from the player's release shot to the basketball rim, g is the gravitational acceleration

exerted on the basketball by Earth's gravitational field, and m is the mass of the basketball. The dotted red lines define the bounds of the basketball's trajectory, and the curved black line represents the center of mass' trajectory.

According to Newton's second law, the ball's force balance, once it's released, is:

$$(1)\ F = ma = mg$$

as shown in Figure 2.
The solution to the force balance gives us the following kinematic equations:

$$(2)\ x(t) = v_x t$$

$$(3)\ y(t) = v_y t + \frac{1}{2} g t^2,$$

where $x(t)$ and $y(t)$ is the horizontal and vertical position of the ball relative to time t, respectively, v_x and v_y are the horizontal and vertical velocity of the basketball at $t = 0$, respectively, and g is the vertical acceleration downwards.

Now, the ideal shot barely skims the front of the rim:

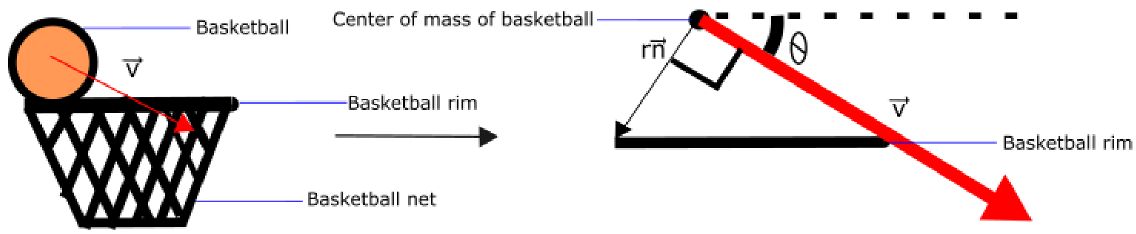

Figure 3. The local motion of the basketball. In this figure, v is the velocity of the basketball at time t, n is the normal unit vector to the ball's velocity, θ is the velocity's angle above the horizontal, and r is the radius of the ball. By multiplying the normal unit vector with the radius, we obtain a vector representing the position difference of the ball's center of mass to the front of the rim.

As shown in Figure 3, we define the unit normal vector to the trajectory as:

$$(4)\ n = \begin{pmatrix} \sin(\theta) \\ -\cos(\theta) \end{pmatrix}$$

$$\Rightarrow (5) \begin{pmatrix} v_x t \\ v_y t - \frac{1}{2} g t^2 \end{pmatrix} + r \begin{pmatrix} \sin(\theta) \\ -\cos(\theta) \end{pmatrix} = \begin{pmatrix} k \\ h \end{pmatrix},$$

where k is the horizontal distance from the basketball's starting position and the rim, h is the height of the rim from the shooter's release position, and r is the radius of the basketball. Note that every unique combination (k, h) represents a unique position in the frame of reference illustrated in Figure 2.

This can be arranged into the following equations:

$$(6)\ r^2 = (v_x t - k)^2 + (v_y t - \frac{1}{2} g t^2 - h)^2$$

$$(7)\ \frac{v_x t - k}{v_y t - \frac{1}{2}gt^2 - h} = -\tan(\theta) = -\frac{v_y - gt}{v_x}$$

After some algebraic manipulation of equations 5 and 6, we have v_x and v_y in terms of t:

$$(8)\ v_x = \frac{(g^2t^4 - 4ght^2 + 4k^2 + 4h^2 - 4r^2)k + 2r(\frac{1}{2}gt^2 - h)\sqrt{g^2t^4 - 4ght^2 + 4k^2 + 4h^2 - 4r^2}}{t(g^2t^4 - 4ght^2 + 4k^2 + 4h^2)}$$

$$(9)\ v_y = \frac{(g^2t^4 - 4ght^2 + 4k^2 + 4h^2 - 4r^2)\left(\frac{1}{2}gt^2 + h\right) + 2rk\sqrt{g^2t^4 - 4ght^2 + 4k^2 + 4h^2 - 4r^2} + 4gr^2t^2}{t(g^2t^4 - 4ght^2 + 4k^2 + 4h^2)}$$

To make a basketball shot more energy efficient, we need to minimize the energy consumption of the shot. The equation for the energy consumption of a basketball shot is:

$$(10)\ E_{translational} = \frac{1}{2}mv^2$$

$$(11)\ v = \sqrt{{v_x}^2 + {v_y}^2},$$

where v is the initial velocity of the basketball. Thus, to minimize $E_{translational}$, we must minimize v^2. And since we want to minimize the final velocity, we get:

$$(11)\ \frac{dv}{dt} = \frac{d(\sqrt{{v_x}^2 + {v_y}^2})}{dt} = 0$$

Solving this equation gives us t, the optimal time it takes for the ball to skim the rim, which can be plugged back into the equation v_x and v_y, giving us the minimum velocity to shoot a basketball.

This final equation is coded in Python available for public use (see Data Availability Statement).

Results

After simplifying a basketball shot from 3-D to 2-D, as shown in Figure 2, and defining the criteria of a successful shot, we deduced two equations modeling the basketball's initial horizontal and vertical velocities based on known constants and time elapsed for the ball to skim the rim. By combining these two equations, we obtained an equation of the ball's initial velocity and used calculus to calculate the optimal time. Plugging the optimal time value into the previous equations of initial horizontal and vertical velocities yields the optimal shot velocity. We took the ball's release height as 2 meters above the ground in the following figures. However, any height can be plugged into the equation to acquire personalized figures on release angle and velocity.

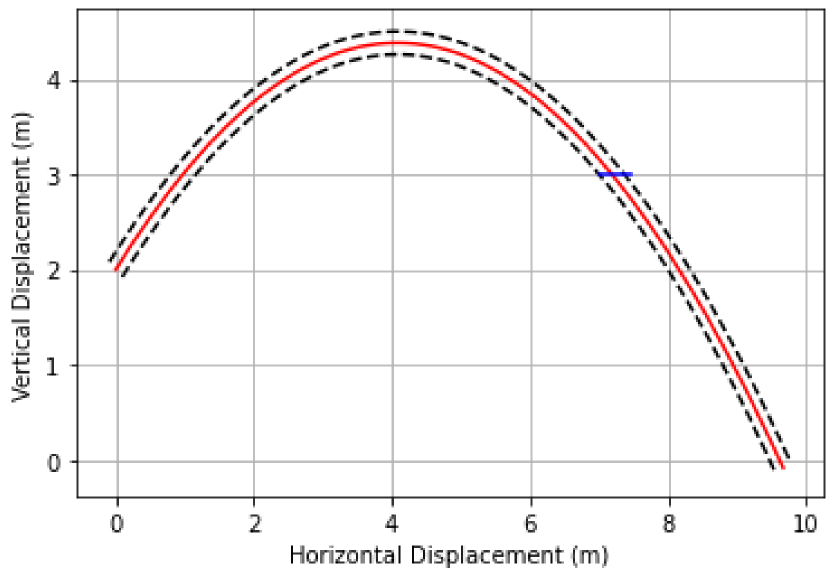

Figure 4. An example of an optimal NBA three-pointer. The red path denotes the trajectory of the center of the basketball, and the dotted lines denote the bounds of the trajectory. The blue line represents the basketball net's rim and is 7 meters away from the player. Note that the ball skims the front of the rim as we expected.

After calculating the optimal velocity, we simulated shots at various ranges (an example shown in Figure 4), confirming that the optimal velocity scores a successful shot for all horizontal distances between 0.5 and 100 meters. However, applications of the optimal velocity equation should be limited to three-point shots because this model doesn't account for air resistance or ball spin. As shown in Figure 5, optimal jump shots in the key take a 78-to-57-degree release angle, mid-range jump shots take a 57-to-50-degree release angle, and three-pointers to deep threes take a 49-to-47-degree release angle.

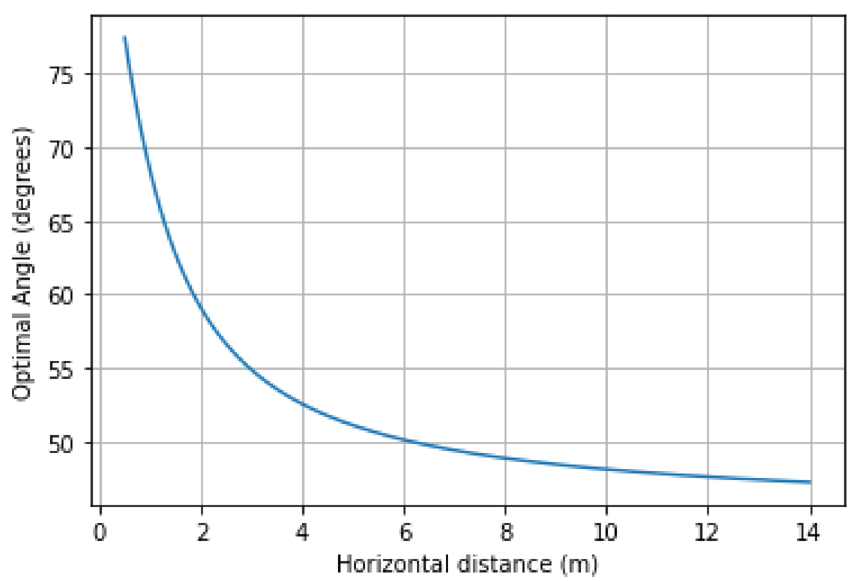

Figure 5. Optimal angle vs. horizontal distance from the net. Note that this is the optimal angle for a two-meter-tall basketball player. The relationship between the optimal angle and the horizontal distance is reciprocal, sharply decreasing as the player initially distances themselves from the net and gradually approaches 45 degrees.

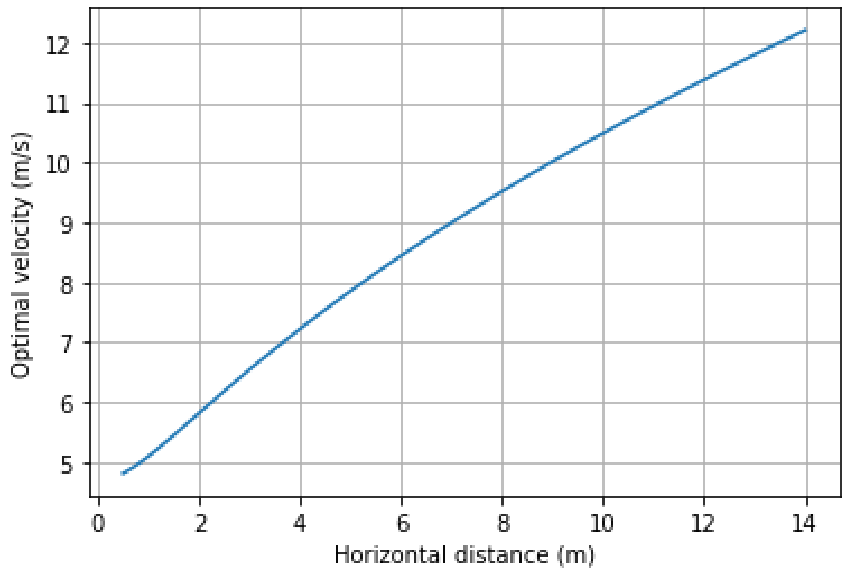

Figure 6. Optimal Velocity vs. Horizontal distance from the net. The relationship between the optimal velocity and horizontal distance closely resembles a linear relationship, implying that the relationship between the minimum energy and horizontal distance is quadratic.

Discussion

We found that the optimal angle of release holds a reciprocal relationship with the horizontal distance from the rim (Figure 5), and the optimal velocity holds a linear relationship with the horizontal distance (Figure 6).

Using the optimal shot velocity, analysts can calculate the efficiency of NBA players' shot efficiency, which serves as another statistic to compare and determine which NBA player is the better shooter. Furthermore, we can compare the efficiency of an NBA shooter with their accuracy to see if a correlation exists between these two statistics. Finally, knowing the energy-optimal velocity for a swish can be used to build a basketball robot with the optimal shooting form.

Conclusion

After we derived the general formula for the most energy-efficient shot, we simulated hundreds of basketball shots to confirm that the equation yields a "successful shot" (a swish) that barely skims the rim. Next, we plotted the relationship between optimal release angle and horizontal distance from the net (Figure 5), which was reciprocal, and the relationship between optimal velocity and horizontal distance from the net, which was closely linear (Figure 6). Players should use the coded equation to familiarize themselves with their personal release angles at the corresponding distance to improve their shooting skills further while being aware of the trade-off between shooting efficiency and accuracy.

Limitations

We realize that basketball players won't be able to reproduce the "ideal" shot with perfect accuracy; however, knowing the qualities that constitute such a shot should allow them to approximate the shot and thus minimize energy consumption to the best of their abilities. Brancazio's study also concluded that knowledge of physics could boost a player's performance. Therefore, players armed with this knowledge should familiarize themselves with shooting at these angles at the corresponding distance to improve their shooting skills further. Furthermore, Brancazio (1982) found that a greater angle of release provides greater leeway (variation in release velocity) to score a shot. However, a greater release angle requires more energy, so players should know the trade-off between shooting efficiency and accuracy.

There are limitations to our study. As mentioned, our calculations did not account for air resistance, which visibly affects basketball shots beyond the three-point line. Moreover, we assumed that the Magnus effect on a spinning basketball was negligible for short shots. Still, the Magnus effect can clearly be seen to alter the trajectories of balls in tennis, ping pong, baseball, cricket, and soccer.

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