HIGH SCHOOL EDITION
Journal of Student Research
Volume 12 Issue 2 (2023)

A First Step Toward Ontic Pluralism in Mathematical Explanation

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ABSTRACT

Although discussions about the nature of mathematical explanation are scarce in the philosophy literature, mathematical explanation plays an integral role in the philosophy of mathematical practice and has important consequences in other branches of philosophy. Various proposals are given to describe the criteria that make certain mathematical proofs more explanatory than others; however, none has been free from objections. These proposals also differ in important ways, which leads to the divergence of the two approaches to mathematical explanation: the ontic approach and the epistemic approach. This paper analyzes the strengths and weaknesses of the popular proposals, defends the ontic approach, and proposes the ontic pluralism account. This new account addresses a significant problem of previous ontic proposals.

Introduction

Compared to the explanation of scientific facts, the explanation of mathematical facts is receiving relatively little attention in philosophy research. However, the explanation of mathematical facts, in addition to its independent interest, has important ramifications in other branches of philosophy.

Mathematical explanation has multiple meanings when put into different contexts. Lyon and Colyvan distinguish two types of mathematical explanations: extra-mathematical and intra-mathematical (2008, p. 3). Extra-mathematical explanation is "explanation in natural science carried out by essential appeal to mathematical facts" (Mancosu, 2008, p. 135). For example, an explanation of a scientific fact that uses a property of prime numbers is an extra-mathematical explanation. Accounts for extra-mathematical explanations need to establish how abstract entities in mathematics connect to the physical world. Intra-mathematical explanation is the explanation of mathematical facts. Such explanations typically appear in the form of proofs, though other forms of explanation are also discussed in the literature.[1] The distinction between proof that merely verifies and proof that explains is widely discussed in the community of mathematicians. This distinction is corroborated by a recent analysis, which concludes that mathematicians routinely describe themselves as explaining mathematics in their research papers (Mejía-Ramos et al., 2019).

This paper focuses on intra-mathematical explanations in the form of proofs. Study of intra-mathematical explanation is important beyond its independent interest. First, accounts for extra-mathematical explanation and scientific explanation benefit from a more complete understanding of intra-mathematical explanation. Second, some theoretical virtues, including explanatory power, are used in some proposals to serve as extrinsic justifications for certain axioms or postulated abstract entities (Maddy, 1992, Ch. 4). For example, if a set of axioms produces more explanatory proofs than another set of axioms, then the difference in the explanatory power may be used to justify the use of the first set of axioms. Third, the intra-mathematical explanation also sheds light on a variation of Quine's indispensability argument (Mancosu, 2008, pp. 140-141). It is possible to

[1] See D'Alessandro, 2020 and Lange, 2018.

develop a parallel of Quine's indispensability argument that justifies the ontological existence of abstract mathematical entities (Mancosu, 2008, pp. 139-140). For example, number theory could be explained using higher-level concepts, and thus, we can postulate these higher-level abstract entities and believe in their existence. This line of argument, albeit the objections against it, is still interesting; a thorough development of this argument requires a more detailed account of mathematical explanations of mathematical facts (Mancosu, 2008, pp. 139-140). We will not delve into the indispensability argument further in this paper.

Understanding the nature of intra-mathematical explanation depends on the answer to a crucial question: what makes a mathematical proof explanatory? This paper summarizes various proposals, highlights the distinction between the ontic and epistemic approach to this question, and gives a unifying account—ontic pluralism.

Popular Proposals

There is no consensus on what makes a mathematical proof explanatory. Various proposals of mathematical explanations have been given by Steiner (1978), Kitcher (2008), Lange (2014), and Inglis and Mejía-Ramos (2021), and others, though each proposal has been unsatisfactory.

Steiner proposes that the best criterion to distinguish an explanatory proof is whether the proof uses the characterizing property of objects or structures referenced in the theorem and whether by varying the characterizing property one can obtain the proof of an analogous theorem. By a characterizing property, Steiner means "a property unique to a given entity or structure within a family or domain of such entities or structures" (Steiner, 1978, p. 143). For example, consider the theorem that $\sqrt{6}$ is irrational. A proof of this theorem utilizes the fact that every positive integer greater than or equal to 2 has a unique prime factorization. In this case, the prime factorization of 6 is a characterizing property of 6. An entity or structure could have multiple characterizing properties when considered in different families of entities or structures. For example, the property that 6 is a unique successor of 5 could also be a characterizing property of 6 in other contexts. Steiner claims that a proof of a theorem is explanatory if and only if it satisfies two criteria. First, the proof must make essential use of the characterizing property of an entity or structure in the theorem. Second, by changing the characterizing property in a family of entities or structures, a proof of a different theorem can be generated.

There are other features of proofs, such as generality, abstraction, and visualization, that have been associated with explanatory proofs. It is also surmised that explanatory proof tends to be those that can lead to the discovery of the theorem. In his paper, Steiner explains the connections between these features and the explanatory value of a proof. First, "generality is often necessary for capturing the essence of a particular, and the same goes for abstraction" (Steiner, 1978, p. 146). Steiner suggests while an explanatory proof is not necessarily general or abstract, an explanatory proof gives the potential for generalization. Taking the above example, one can use a similar method to prove that for any positive integer n, $\sqrt{n}$ is rational if and only if n is a perfect square. Although the proof of the irrationality of $\sqrt{6}$ is not in itself general in the sense that it does not directly prove the irrationality of other positive integers, the method can be adapted easily to prove more general results. Therefore, this explanatory proof gives the potential for generalization. Second, explanatory proof can be used to discover results because proofs that qualify as being explanatory involve the characterizing property of the content of the theorem. Deforming such quality directly gives more results and leads to discovery. Finally, a characterizing property can be a geometric property. Hence, visual proof can, but not necessarily have to, be explanatory.

Several objections are raised against Steiner's proposal[2], the most notable of which is a geometric theorem used by Lange as a counterexample (2014). The theorem states that "if ABCD is an isosceles trapezoid as shown in figure 1 (AB parallel to CD, AD = BC) such that AM = BK and ND = LC, then ML = KN" (Lange,

[2] See Resnik & Kushner, 1987, pp. 145-151.

2014, p. 501). Two proofs of a proposition about isosceles trapezoids are given, and one is more explanatory than the other. The only characterizing property of the mathematical entities or structures in the theorem is the unique property (AD = BC) that makes a trapezoid isosceles, but one can vary this characterizing property only by considering the same proposition on non-isosceles trapezoids. This new condition does not yield a new proposition; hence, Steiner's proposal fails to account for the more explanatory proof. I think that Lange's example gives an effective objection because making the trapezoid non-isosceles is more than a deformation of characterizing property. It is also a generalization: the set of isosceles triangles is a subset of the set of triangles. Hence, Steiner's proposal needs to be revised to account for the range of legitimate deformations on the characterizing property.

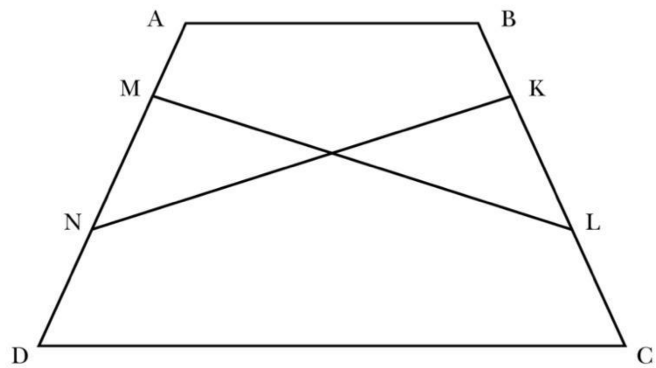

Figure 1. An isosceles trapezoid. From. "Aspects of mathematical explanation: Symmetry, unity, and salience," by M. Lange, 2014, *Philosophical Review*, *123*(4), p. 501.

Kitcher offers a different account of mathematical explanation. A concise summary of Kitcher's account can be found in Hafner and Mancosu's paper (2008). According to Kitcher, our understanding of the world is advanced if the number of patterns of argument used to derive knowledge decreases. In other words, science and mathematics progress when the same patterns of arguments can be used again and again to derive more conclusions. Therefore, the highest goal of mathematical explanations is to provide such unification. Given a proof, one can distill an argument schema and a filling instruction such that one can reconstruct the proof by filling the right words to the argument schema according to the filling instruction. Kitcher judges the explanatoriness of a proof based on how many valid explanations are stored within the argument schema distilled from the proof. The degree of unification is directly proportional to the size of the set of conclusions and inversely proportional to the size of the set of argument patterns. Kitcher's explanation, different from Steiner's proposal, sees explanation as a global phenomenon. The proofs are not judged individually; instead, they are put within a broader context of systematization.

Hafner and Mancosu's paper challenges Kitcher's proposal by comparing three proofs of a theorem T in the theory of real closed field. Proof I uses an algorithm that outputs the truths of any theorem in the theory of a real closed field. General as it is, few mathematicians use the algorithm because it involves laborious computation and does not explain the theorem. Proof II uses a transfer principle, which states that any theorem that is true in the field of real number (a special case of real closed field) is true in any real closed field. Proof III proves the theorem T using results from algebraic geometry. Mathematicians tend to choose the third proof instead of the second proof because they are interested in seeing how the axioms of real closed field explain theorem T. Kitcher's model, if succeeds, should be able to output the third proof. However, the model actually concludes that proof I is the most explanatory because its argument pattern can be used to prove all the results in the theory of real closed fields. Hafner and Mancosu also provide a detailed account of why proofs II and III are incomparable under Kitcher's framework. As a result, Kitcher's model fails as a tool to compare explanatoriness. One reason why the model fails is that it simply assumes that explanatory power can be compared quantitatively, and it overlooks the qualitative aspects of the proofs.

Lange further challenges Kitcher's idea by claiming that "a new proof technique can explain why some theorem holds even if that technique allows no new theorem to be proved" (Lange, 2014, p. 523). He refutes the characterization of mathematical explanation as a global phenomenon and insists the explanatory value of proofs be judged within the proof itself.

As opposed to appealing to characterizing properties or unification, Lange (2014) proposes that what distinguishes one proof from another as being explanatory is the use of a symmetry that is salient in the result. The symmetry here can be geometric as well as algebraic. For example, A theorem that involves the bilateral symmetry of an isosceles triangle has a geometric symmetry, while a theorem that involves the operation-preserving properties of complex conjugation has an algebraic symmetry. Suppose a proposition is discovered, and it exhibits certain kinds of symmetry. Lange states that while a non-explanatory proof merely shows that the result is true through some brute-force manipulation, an explanatory proof is able to account for the symmetry of the result by tracing the symmetry back to a similar symmetry in the problem statement.

For example, Lange's proposal succeeds in accounting for the explanatoriness of the counterexample to Steiner's proposal. The explanatory proof of the theorem uses the bilateral symmetry of the isosceles trapezoid. The symmetry in the conclusion surprises us, and through the proof, one can see that this symmetry is traced back to the symmetry in the bilateral trapezoid, an object in the setup. The proof is explanatory because it relates the symmetry in the setup to account for the symmetry in the conclusion.

Lange further develops that symmetry is just an instance of a broader account of mathematical explanation. Suppose a proposition is discovered, and it exhibits certain salient features, and thus, demands explanation. A proof is explanatory if it uses a similar feature in the setup, i.e., if it traces the salient feature of the result back to the feature of the theorem. Examples of such features include symmetry, unity, and simplicity. Symmetry is the symmetry in a mathematical sense explained above. When a result holds for multiple cases at the same time, or cases of different kinds, then a proof that accounts for the cases all at once exhibits unity. When the result is articulated in simple terms, a proof that does not use complicated mechanisms, such as cumbersome algebraic operations, and uses only the simple objects in the setup instead, exhibits simplicity.

An objection to this proposal is given by D'Alessandro, who claims that Lange's proposal fails to account for the case when a conjecture is used to explain a fact. He gives an example of $P \neq NP$ conjecture. $P \neq NP$ states that not every problem to which an attempted answer could be verified in polynomial time has an algorithm that solves the problem in polynomial time. If $P \neq NP$ true, many results can be explained. For example, Chvatal in 1979 showed that one of the algorithms for the Set Cover problem returns a cover of size of at least $(ln\ ln\ n) \times OPT$. OPT is the greatest lower bound if $P \neq NP$ is true. Hence, $P \neq NP$ explains the greatest lower bound. However, there is no proof of $P \neq NP$, so no saliency connection can be established between the results and the proofs (D'Alessandro, 2020, pp. 588-590).

I think this objection is unsound. First, the fact that $P \neq NP$ can prove that OPT is the greatest lower bound does not guarantee explanatoriness. There is a distinction between proofs that merely prove and proofs that explain, so D'Alessandro needs to show how $P \neq NP$ explains OPT. Second, D'Alessandro is thinking of an explanation of OPT through the conjecture itself, but Lange's account only applies to explanation in the form of proofs. When one says that "the conjecture explains the theorem," then one can construct a proof of the theorem that assumes the truth of $P \neq NP$. It is true that no proof of $P \neq NP$ is available. But one does not need to prove $P \neq NP$ in order to show the relationship between OPT and $P \neq NP$. So long a proof of OPT that assumes the truth of $P \neq NP$ is available, one can find the salient features of OPT and trace such a feature to a similar feature in the proof.

D'Alessandro gives another example. He claims that Hadwiger conjecture, if true, can explain the four-color theorem; but there is no proof of Hadwiger conjecture, so one cannot trace back the salient features of the four-color theorem to a similar feature in the proof (D'Alessandro, 2020, pp. 588-590). This example does not give a valid objection for the same reason. One can work the explanatory argument of the four-color theorem into a proof that assumes the truth of the Hadwiger conjecture. With this proof, one can analyze how the salient features of the theorems are traced to a similar feature in the setup.

In what follows, I will give two counterexamples of Lange's proposal.

First, consider the classic proof of irrationality of e (Rudin, 1976, p. 65). Suppose for the sake of contradiction that $e = \frac{p}{q}$ for some positive integers p, q. Consider the Taylor series expansion of e:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Then,

$$\frac{p}{q} = \sum_{n=0}^{q} \frac{1}{n!} + \sum_{n=q+1}^{\infty} \frac{1}{n!}$$
$$\frac{p}{q} - \sum_{n=0}^{q} \frac{1}{n!} = \sum_{n=q+1}^{\infty} \frac{1}{n!}$$

Multiply both sides by $q!$, we get

$$p(q-1)! - \sum_{n=0}^{q} \frac{q!}{n!} = \sum_{n=q+1}^{\infty} \frac{q!}{n!} = \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \cdots$$
$$< \frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \cdots = \frac{1}{q}$$

The left side is an integer, but the right side is between 0 and 1, a contradiction. Hence e is irrational.

This proof is explanatory. The Taylor series expansion explains why e is irrational. Here, Steiner's account gives a clear reason for the explanatoriness of the proof. The Taylor series expansion of e used in this proof is the characterizing property of e. By varying the characterizing property, one can obtain another theorem. For example, one can use essentially the same method to prove that $cos\ cos\ 1$ is irrational. Hence, the two criteria in Steiner's account are met. However, Lange's proposal fails to account for the explanatoriness of the proof. There is no feature that stands out in the setup of the theorem, which simply says the mathematical object e has the property of being irrational. As a result, no correspondence can be drawn between the salient features of the setup and the same features in the proof.

Now consider another example. Cantor proves that rational numbers are countable, i.e., there is a one-to-one correspondence between the set of rational numbers and the set of natural numbers. One way of proving this theorem is to prove that the set of positive rational numbers is countable. The set of positive rational numbers can be defined as:

$$\left\{\frac{p}{q} : p, q \in Z^{+}, gcd\ (p, q)\ = 1\right\}$$

First, we list the set of positive rational numbers in a two-by-two array. The i^{th} row contains all and only the irreducible fractions with denominator i.

$$\begin{array}{l} \frac{1}{1}\ \frac{2}{1}\ \frac{3}{1}\ \frac{4}{1}\cdots \\ \frac{1}{2}\ \frac{3}{2}\ \frac{5}{2}\ \frac{7}{2}\cdots \\ \frac{1}{3}\ \frac{2}{3}\ \frac{4}{3}\ \frac{5}{3}\cdots \\ \frac{1}{4}\ \frac{3}{4}\ \frac{5}{4}\ \frac{7}{4}\cdots \end{array}$$

Start counting from $\frac{1}{1}$. Then $\frac{1}{2}$ and $\frac{2}{1}$. Then $\frac{1}{3}$, $\frac{3}{2}$, and $\frac{3}{1}$. Continue counting in diagonal lines in this fashion. It is clear that every positive integer is counted exactly once in this manner. Hence, there is a one-to-one correspondence between the set of positive rational numbers and the set of natural numbers, so the set of positive rational numbers is countable.

This proof is explanatory because it gives a visual idea of how to count rational numbers. Again, Steiner's characterizing property is successful in accounting for its explanatoriness. The characterizing property of the set of positive rational numbers is that it can be represented as a fraction, which is essentially an ordered pair of two positive integers. The proof above uses such characterizing property by giving an intuitive idea of how to count such ordered pairs using diagonal lines. Moreover, varying the characterizing property does generate new theorems. For example, if one changes the set of ordered pairs of two positive coprime integers to the set of ordered pairs of three positive coprime integers, then a similar enumeration can be applied. One only has to construct a three-dimensional grid and make a similar argument. Instead of counting by diagonal lines, one counts by diagonal planes.

Lange's proposal fails to account for the explanatoriness in this example for a reason similar to the reason why the proposal fails to account for the first example. There is no salient feature in the setup of the problem, so no saliency correspondence could be drawn. Like the first example, this example suggests a general problem with Lange's proposal. If a theorem is in the form object O has property P, then a proof of such a theorem may be explanatory because it uses the characterizing property of O. Instead of any salient features, which can hardly be found in theorems similar to the two examples, the characterizing property of O explains P. Therefore, Steiner's proposal works well with this kind of proposition, while Lange's proposal fails.

A proposal drastically different from Steiner's, Kitcher's, and Lange's is given by Inglis and Mejía-Ramos (2021). According to them, Wilkenfeld's functional explanation proposal can be integrated into the proposal in mathematical explanations. The basic premise is that explanations must generate objectual understanding, a type of understanding that is distinct from propositional understanding. Propositional understanding is understanding of whether a proposition is correct. For example, the grasp of the statement "there are two cows on the field" generates propositional understanding. Objectual understanding is understanding of an object. For example, the knowledge of physics or the knowledge of trees gives objectual understanding. Objectual understanding is often more holistic and admits a degree of understanding. Every mathematical proof supplies propositional understanding because it verifies the truth of a proposition, but only certain mathematical proof generates objectual understanding.

Inglis and Mejía-Ramos define that a proof is explanatory if and only if the proof generates objectual understanding. They define that the objectual understanding of some phenomenon is maximal if it yields "fully comprehensive and maximally well-connected knowledge" about the phenomenon, and the objectual understanding is greater if it is less distant from the maximal understanding (Inglis & Mejía-Ramos, 2021, p. 6377).

To assess the degree of understanding produced by a proof, a relatively simple and uncontroversial model of the human mind is used (see figure 2). A schema is a cognitive structure that allows people to treat "multiple elements of information as if it were a single element" (Inglis & Mejía-Ramos, 2021, p. 6378). For example, despite the complicated composition of a tree, one can immediately identify a tree when one sees the tree because of the schema. The model asserts that sensory input, which can be stored in a very short term, is processed in the sensory memory to be selected into the working memory, where cognition and thinking take place. Working memory can input information from sensory memory but also from the schema stored in the long-term memory. Working memory has a small capacity limit, but long-term memory can store many schemas. The end-product of the working memory processing is encoded into the long-term memory to form a new schema.

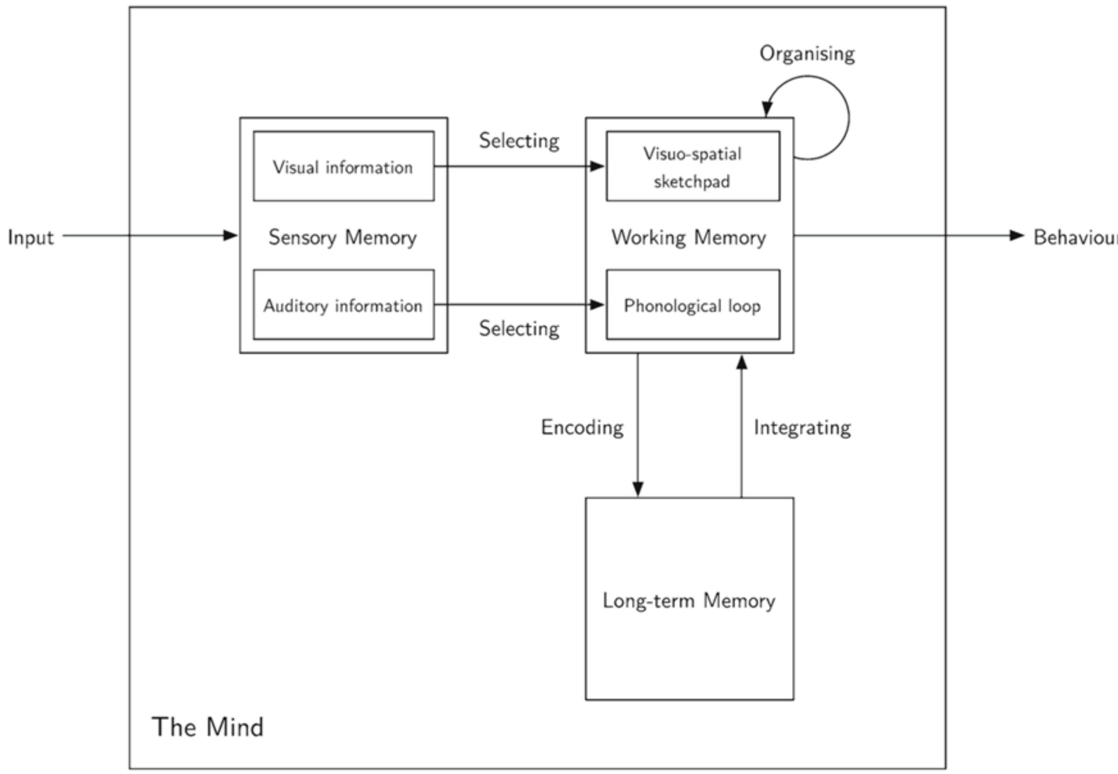

Figure 2. A modal model of the mind. From "Functional explanation in mathematics," by M. Inglis & J. P. Mejía-Ramos, 2021, *Synthese*, *198*(26), p. 6379.

Understanding is the process of integrating new schema into the long-term memory. This process can be done in multiple stages: 1) select from sensory memory; 2) process in the working memory; 3) integrate existing schemas from long-term memory; 4) re-organize this knowledge into a new schema. Working memory also contains two subsystems: visuospatial sketchpad and the phonological loop. Processing information in these two subsystems could enhance the limited capacity of the working memory. Hence, a proof is explanatory if it satisfies at least three properties: 1) it "makes it easy to select the information from sensory memory into working memory;" 2) it makes it easy to extract existing schemas from long-term memory and build connections and integrate new information with the existing schemas; 3) it makes it easy to split working memory load into the visuospatial sketchpad and phonological loop (Inglis & Mejía-Ramos, 2021, p. 6381).

Inglis and Mejía-Ramos claim that Steiner's characterizing property, Kitcher's unification, and Lange's saliency can be incorporated into this model. First, the unification of argument patterns makes it easy to integrate different schemas; hence, it contributes to the explanatoriness. Second, the use of a visuospatial sketchpad in splitting the working memory load accounts for the fact that visual proof tends to be more explanatory. Third, if a proof uses the characterizing property or the salient feature of the proposition, then it makes it easy to extract existing schemas from long-term memory. This is because a proof using the characterizing property makes it easy to compare objects that have characterizing properties or objects that do not.

I think the main problem with this proposal is that it is oddly general. It does not directly address the problem of what specific features of the proofs contribute to the explanatoriness. Further contemplations on how the proposal works on specific examples and how it interprets other proposals illustrate the problems of the proposal in practice.

First, this proposal does not seem to work well with Steiner's as claimed. According to Steiner, the existence of a characterizing property is a criterion to judge explanatoriness. This does not imply that a subject reading an explanatory proof necessarily notices the characterizing property and compares the objects that have or do not have the property. Nor will the subject attempt to deform the characterizing property to generate a new theorem. Doing so may indeed make it easy to extract schema from the long-term memory, but the necessity of such behavior is not implied. Therefore, the proposal fails to account for the plausibility of Steiner's proposal. If one uses Inglis and Mejía-Ramos' proposal seriously, one needs to demonstrate precisely how long-term memory is extracted and how new schemas can be incorporated.

Second, Inglis and Mejía-Ramos give a proof of the infinitude of prime number: "Assume that there is a largest prime, p. Consider the number one greater than the product of all the primes: $n = 2 \times 3 \times 5 \times \ldots \times p + 1$. Either n is a product of primes or it is a prime larger than p. The latter would contradict our premise, so n must be a product of primes. But if n is a product of primes and has no prime factors greater than p, then one of its factors, q, must be in the sequence $2,3,5, \ldots, p$, and therefore divides the product $2 \times 3 \times 5 \times \ldots \times p$. However, since it is a factor of n it also divides n. But a number which divides two numbers also divides their difference, so q must also divide $n - (2 \times 3 \times 5 \times \ldots \times p) = (2 \times 3 \times 5 \times \ldots \times p + 1) - (2 \times 3 \times 5 \times \ldots \times p) = 1$. However, no prime divides 1 so q is not in the sequence $2,3,5, \ldots, p$. It follows that if n is composite, it has at least one factor greater than p. This is a contradiction. Therefore, there is no largest prime number; there are infinitely many primes" (Inglis and Mejía-Ramos, 2021 pp. 6383-6384).

According to Inglis and Mejía-Ramos, this proof is non-explanatory and does not generate objectual understanding. However, this proof, in fact, does generate objectual understanding of prime numbers. Readers of this proof acquire an understanding as to what happens when there are only finitely many primes. They will also understand the process of generating a new prime in a world where there are finitely many primes. This new knowledge moves the readers closer to the maximal understanding of prime numbers. This example shows that Inglis and Mejía-Ramos' relatively flexible and general criterion sometimes identifies some non-explanatory proofs as explanatory ones.

Ontic and Epistemic Approach to Mathematical Explanation

Inglis and Mejía-Ramos's proposal is different since it is an example of the epistemic approach to mathematical explanations as opposed to the ontic approach. The distinction between the epistemic and the ontic approach in the context of mathematical explanation is first given by Delarivière et al (2017), who borrow this distinction from Salmon (1984) on a similar debate in the context of scientific explanation. The ontic approach sees explanations as "exhibitions of the ways in which what is to be explained fits into natural patterns or regularities", while the epistemic approach sees explanations as statements or arguments (Salmon, 1984, p. 293). The ontic approach treats a proof as an object whose explanatory value can be judged using certain objective criteria, while the epistemic approach sees a proof as a way of communication or representation. An account of explanation is ontic if it implies the statement that there is no proof that can be seen as explanatory in the view of one person but not explanatory in the view of another person. An account is epistemic if it implies that there is, at least in principle, such a proof. The ontic approach does not deny that explanatory proof tends to generate understanding, but while the epistemic approach takes the generation of understanding as an essential criterion for explanatoriness, the ontic approach only takes the increase of understanding as a consequence.

I prefer the ontic approach over the epistemic approach because it is more direct. In the discussion of the explanatory value of mathematical proof, it is more natural to analyze the property of the proof itself and explain how the generation of understanding becomes a consequence of explanations. In other words, the ontic account has the benefit of offering a closer look at the properties of the proofs themselves. In the next section, I will develop a pluralism account based on the ontic approach.

The ontic approach faces several objections. Delarivière et al. (2017) suggest that "a mathematical proof can be seen as an argument by which one convinces oneself or others that something is true" (p. 311). It is plausible to think so because proofs are used to communicate between mathematicians so that a mathematician could understand the fruit of another mathematician's work. Since communication through proofs is used by the explainers to generate the audience's understanding, an account of mathematical explanation will interpret how such effective communication is achieved. Therefore, the assumption that a proof is an argument leads to the conclusion that an epistemic reading of a mathematical explanation is a more natural choice.

I think that a proof should be treated as a representation of mathematical content. One can change the specific expressions of a proof while not making changes to the content it contains. The same content can be written in a way that is accessible to those with little background or in a way that is more technical and rigorous. The relationship between a proof and its content is analogous to the relationship between a token and a type. For example, while a token of "9" could be a written mark "9" on the paper, a letter "9" on the screen, an inscription "9" on a stone, the object "9" itself is the type.

The written proofs are essential to the communications between mathematicians; however, we are interested in the explanatory value of the content contained in proofs instead of the written proofs themselves. Perhaps only some people may be able to grasp a subset of the information contained in a proof, but the explanatoriness of a proof can be judged from the overarching set of information that it contains regardless of what subset of information a person can grasp. In this reading, the ontic approach is a more natural choice.

Inglis and Mejía-Ramos (2021) present another challenge to the ontic approach. They argue that the ontic approach is based on a false assumption that mathematicians' "judgments of explanatoriness are likely to coincide" (p. 6374). They imply that since such judgments do not coincide, it is not meaningful to speak of explanation as an object. I suggest that treating an explanation as an object does not contradict the fact that the judgments made by mathematicians do not coincide. Mathematicians, and philosophers alike, who concern about the topic of explanations, are inquirers who attempt to understand the objective explanatory values of proofs. Although mathematicians' intuitive judgments are powerful indications of the objective explanatory value of a proof, these judgments do not determine such value.

It is worth noting that treating mathematical explanations as objects does not imply that the question of whether proof P is explanatory as a determinate truth value. If two mathematicians disagree on the explanatory value of proof P, it is likely that the proof is explanatory *in some ways* but not in others. For example, a theorem T may exhibit the salient features of symmetry and simplicity (in Lange's sense). Suppose the proof P proves T without heavily using computations and algebraic manipulations. This proof traces the feature of simplicity from the conclusion back to the setup through the proofs but not the feature of symmetry. Two mathematicians who place different weights on symmetry and simplicity may reach different conclusions on whether P is an explanatory proof.

Explanatory values should be treated as qualitative descriptions of a proof instead of a quantity associated with a proof. Hence, it is not necessary for the explanatoriness of every proof to have a determinate truth value.

Ontic Pluralism

There is no consensus as to whether the ontic approach or the epistemic approach is better. The following section will be built on an ontic interpretation of mathematical explanation, and my new proposal attempts to address a problem in the current ontic proposals.

In the existing ontic proposals, a model accounting for mathematical explanation is described, and it is tested against some typical mathematical proofs. The problem with this method is that there are numerous branches in mathematics, and within each branch, there are numerous significant results. The richness of mathematical knowledge and the lack of representative theorems and proofs make it hard to account for all cases with a single proposal. This difficulty is supported by the fact that many arguments against the proposals are based on counterexamples.

There are two ways to address this central issue. First, a new proposal of mathematical explanation can be given, which is robust enough so that there is no room for any counterexample. Second, one argues for pluralism on the ontic proposals, which leaves room for new proposals when counterexamples are found. So far, no single proposal seems to be free from objections. Therefore, I suggest a pluralism on the ontic proposals. In what follows, I will first develop an overarching framework under which ontic proposals can be admitted into and compared in the pluralism account. Then, I will argue that each ontic proposal discussed in section II contributes to the pluralism account. Lastly, I will give examples to show how these ontic proposals can be compared.

First, I define an ontic proposal to be valuable if and only if it can be incorporated into the pluralist account based on the following criteria.

Any valuable ontic account of mathematical explanation needs to exhibit *how* an explanatory proof causes “what is to be explained fits into the natural patterns or regularities” (Salmon, 1984, p. 293). A strict definition of the term “natural patterns” and “regularities” will not be given here because such a definition will narrow down the range of valuable proposals and potentially favor certain proposals over others. For a proposal to be valuable, it should give clarification of what it means for mathematical patterns to be natural and regular by answering the “how” question. For example, suppose that a proposal suggests that the explanatory value of a proof is directly proportional to its generality and use of abstraction. This proposal can be considered valuable if and only if the proponents of such proposals can demonstrate precisely how abstraction and generality contribute to explicating the natural patterns in what is to be explained.

Having set an overarching criterion for what an ontic proposal needs to have in order to be admitted into the pluralism framework, the next step is to provide a way to evaluate and qualify the proposals in the framework and to add new ontic proposals. Every proposal fails to account for explanations in some cases. Therefore, one needs to specify the ranges in which various ontic proposals are successful. The ranges certainly have overlapping areas, since some explanatory proofs are successfully accounted for by multiple proposals. These overlapping areas make my account for mathematical explanations truly pluralist.

The relative value and range of applicability of valuable ontic proposals within the pluralism framework are determined by their success or failure in judging the explanatory value of proofs.

Consider an explanatory proof P, to a theorem T, and two valuable ontic proposals. If both proposals give a strong reason that P is an explanatory or non-explanatory proof, then the relative value of both proposals increases. It is not necessary to favor one over another because both proposals shed light on why X is explanatory.

If, however, proposal A concludes that P is explanatory and proposal B concludes that P is non-explanatory, then two cases should be considered.

First, one of the proposals gives a stronger reason for its conclusion about the explanatory value of P than the other proposal. Without loss of generality, suppose that proposal A successfully accounts for why P is explanatory, while the reason given by proposal B for why P is non-explanatory is not as strong. Then, the relative value of proposal A increases. If there is a set of proofs S, which contain P, such that proposal A can account for the explanatory value of every member of S in a way similar to how it accounts for the explanatory value of P, then we can say that proposal A applies to S. If there is a set of proofs S', which contains P, such that proposal B fails to account for the explanatory value of every member of S' in a way similar to how it fails to account for the explanatory value of P, then we can say that proposal B does not apply to S'. In this way, we know more about the range of applicability of the two proposals.

Second, though the two proposals disagree with each other, both succeed in showing how P fits what is to be explained into natural patterns or regularities. This disagreement is an indication that the explanatoriness of P does not have a determinate truth value (see the above section for why such a conclusion can be accepted).

If none of the proposals in the pluralism framework fails to give a strong reason describing the explanations or the lack of explanations provided by a proof, then a new proposal should be introduced to account for this proof. The flexibility of adding proposals makes the ontic pluralism framework robust to counterexamples.

Every ontic proposal discussed in the "popular proposal" section is valuable. Steiner (1978) claims that a mathematical proof of a theorem T is explanatory if it satisfies two criteria. First, the proof uses a characterizing property of a mathematical object or structure in the theorem T. Second, there exists a family of theorems such that T is a member of this family, and by "deforming" the characterizing property of T, one can prove another result in the family of propositions. Here, the proposal specifies that what makes the patterns in theorem T natural is the characterizing property. The patterns in the family of mathematical propositions become natural through the explanatory proof because they are treated as the consequence of the characterizing property and the variations of the characterizing property.

Kitcher (2008) claims that a proof of a theorem T is explanatory if the argument patterns used in the proof contribute to unification. The more conclusions this argument pattern can draw, and the fewer argument patterns the proof of T need, the more unification is attained. It is tempting to treat Kitcher's proposal as an epistemic one. This is because Kitcher sees the value in explanation as advancing our understanding of the world, and he compares the explanatory value of a proof by looking at the argument patterns. The emphasis on understanding and the treatment of proofs as arguments make Kitcher's proposal seem epistemic. However, Kitcher's proposal has more ontic elements. First, Kitcher's unification answers the "how" question by addressing how natural patterns and regularities are attained through explanatory proofs. According to Kitcher, if the proof of T is explanatory, then it is interpreted as a member of an optimal systematization, which provides organization to the scattering mathematical results. Therefore, the proof of T fits T into a natural member of the optimal systemization. Second, the argument patterns described by Kitcher can be treated as properties of the content of mathematical proofs. Interpreted broadly, two proofs have the same content if and only if they use the same argument pattern. The argument patterns and the unification they provide can be regarded as objective features of proofs. Hence, Kitcher's proposal is a valuable ontic proposal.

Finally, Lange (2014) claims that a proof of a theorem T, which consists of a setup and a conclusion, is explanatory if the proof traces a salient feature of the conclusion of T back to a similar feature in the setup of T. Such feature includes symmetry, unity, and simplicity. Here, the proof makes the salient feature in the conclusion less of a mathematical coincidence but more of a direct consequence of the similar feature in the setup. Mathematical coincidence is the opposite of natural patterns because it is the result of brute computations. By tracing the salient feature of the conclusion back to the setup, the explanatory proof accounts for how seemingly coincidental results are natural. Hence, Lange's proposal is also valuable.

Having specified the details of my pluralism account, I will demonstrate how proposals within this account can be evaluated by showing concrete examples.

In Lange's paper on mathematical explanation (2014), Lange gives a counterexample against Steiner's proposal (see section II). The counterexample illustrates a key issue of Steiner's proposal. Suppose an object O may or may not have property Q, and suppose a theorem T has the following form: statement X about O is true if and only if O has property Q. Assume further that Q is the only characterizing property, and that no other theorem is produced when Q is false. Let S' be the set of proofs to the theorems in this form. In the counterexample described in section II, O is a triangle, Q is the property that O is isosceles, and X is the statement that includes the setup of the diagram and the conclusion ML = KN. "Being isosceles" is the only characterizing property of the triangle, and varying the property obliterates X and any other similar theorems. Therefore, the proof to the theorem in the counterexample is a member of S'.

Steiner's account does not succeed in giving reasons for the explanatory values of every proof in S'. This is because the mere fact that there are no similar theorems when the property Q is varied does not directly imply that the proof is non-explanatory. In other words, the explanatory value of any *proof* of theorem T should not be affected by the fact that there simply does not exist a family of *theorems* that can be reasonably associated with the deformation of property Q. Although deforming the characterizing property does not yield new theorems as required by Steiner's proposal, a proof can still achieve explanations in other ways, such as by establishing a clear link between property Q and theorem T. Therefore, Steiner's account falters on S'.

Lange's proposal succeeds in accounting for the explanatoriness of the counterexample he gives. This proposal describes clearly how the use of symmetry explains the geometric theorem. Suppose T is a geometric theorem involving a symmetry in its conclusion, and this symmetry is a salient feature of T. Let proof P be a proof that traces the symmetry in the conclusion back to the setup of T such that the symmetry in the setup accounts for the symmetry in the conclusion. Let S be the set of proofs in the form of P to theorems in the form of T. Then, Lange's proposal successfully accounts for the explanatory values of all proofs in S. Therefore, the range of applicability of Lange's includes S.

My counterexamples to Lange's proposal also help delineate the range of applicability of Lange's and Steiner's proposal within the pluralism framework (section II). Lange's proposal fails to analyze the explanatory value of the classic proof of the irrationality of *e* and Cantor's proof of the countability of rational numbers. Moreover, it fails to account for the proof of the theorems in the form object O has the property P (section II). Let S' be all proofs in this form. Then, Lange's proposal does not apply to S'. In section II, I have also explained how Steiner's proposal applies to the proofs in S'. Hence, through the counterexamples to Lange's proposal, the range of applicability of both proposals becomes more precisely described.[3]

Possible Objections

My ontic pluralism account does not subject to the common objections to pluralism.

First, pluralism is often criticized as implying that "anything goes." It is clear that my account does not share this issue. An ontic proposal can be considered and integrated into the pluralism account if and only if it answers the "how" question described above. There is a hard limit that distinguishes valuable proposals from other proposals. Additionally, different ontic proposals are tested against mathematical proofs, which distinguish one proposal from another in terms of the range of applicability. Therefore, the proposals within the pluralism framework do not have the same value.

Second, pluralism is often criticized as bringing and comparing incompatible things within a single account. In my case, it may be argued that some proposals, such as Kitcher's, are too different from others such

[3] It is worth noting that Delarivière et al. (2017) sketches an epistemic pluralism account. While both his account and mine are pluralist, there are two important differences. First, although Delarivière et al. are selective in which proposal to include in their pluralism account, and they test the proposal with cases of proofs, they do not provide a way to clearly describe the range of applicability of different proposals with the proofs. Second, I favor the ontic approach. The relative merit of the two approaches has been discussed in section III. Hence, the strategy of interpreting the proposals within the pluralism account is necessarily different.

that they cannot be accepted at the same time. My proposal addresses the issue by having an overarching criterion. However different proposals may be, the extent to which they contribute to mathematical explanations can be compared in terms of how well they answer the "how" question. Instead of bringing contradictory elements to the pluralism account, the diversity of the proposal adds to the understanding of mathematical explanations. For example, the explanatory value of a proof can be evaluated both locally with Steiner's and Lange's proposal and globally (i.e. in the context of a systematization) with Kitcher's proposal. This way of assessing a proof is not contradictory. One may argue that the unifying criterion makes my account monist in nature. I suggest that although the "how" question criterion is important in my proposal, the pluralism aspect of my account comes from the multiple ways in which the broad criterion is met. The criterion is monist, but the way to achieve this criterion is pluralist.

One may feel uncomfortable about the idea that the values of the ontic proposals seem to be put above the values of mathematicians' intuitive judgment. This concern is reasonable since many previous accounts use mathematicians' judgment as a benchmark for their own proposals. However, I suggest that mathematicians' judgment and the reasons they give form a proposal that can be compared the same way as other proposals within my account. If a mathematician claims that a proof is explanatory based on a subjective feeling, then his or her judgment will not be considered. But if a mathematician gives a strong reason as to why a proof is explanatory, and his or her judgment differs from all ontic proposals in the pluralism framework, then the mathematician's reason can be treated as a proposal itself, whose range of applicability can be sketched by testing against other proofs.

Conclusion

An ontic pluralism account of mathematical explanation can enjoy the benefit of the ontic approach while being robust against potential counterexamples. Since the debate on the ontic and epistemic approach will likely not settle soon, my proposal gives a more complete understanding of mathematical explanations by addressing the shortcoming of existing ontic proposals, namely, the vulnerability against counterexamples. Under the pluralism framework, counterexamples do not detract from the values of individual ontic proposals; instead, they provide knowledge about their precise range of applicability.

The next valuable topic of investigation would be to enrich the pluralist account through more examples and qualifications of existing proposals. It is also fruitful to explore the far-reaching consequences of an increasingly sophisticated understanding of intra-mathematical explanations discussed in the introduction of this paper. These consequences include a better understanding of extra-mathematical explanation and scientific explanation, a possible extension into an ontological argument, and a possibility of explanations as an extrinsic justification of mathematical axioms.

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