HIGH SCHOOL EDITION
Journal of Student Research

Volume 12 Issue 2 (2023)

# Investigating Convergence and Divergence of Sequences

Jayan Gandhi[1] and Hayk Sedrakayan[#]

[1]Buckingham Browne and Nicholas
[#]Advisor

## ABSTRACT

Convergence and divergence questions and limit of sequences appear in many different levels of mathematics including high school calculus and college level research topics. In this paper, we are providing different approaches and techniques to deal with limit of sequences, convergence and divergence. We also consider several applications and different examples to explain how these techniques can be used.

## Introduction

In this section we consider some problems related with the limit of sequences and we study the existence of the limit of a sequence. Recall that a sequence is called convergent if it approaches some finite limit, otherwise it is called divergent. Example 1. Let $x_n$ be a sequence of positive real numbers, such that $x_1 = 2022$ and is $x_n$ a convergent or divergent sequence?

$$x_{n+1} = \frac{2022 + 337x_n}{337 + 3x_n}.$$

Answer. Let us assume that $L$ is the limit of the sequence, then we have that

$$L = \frac{2022 + 337L}{337 + 3L}.$$

Multiplying both sides by $337 + L$, we obtain that
$337L + 3L^2 = 2022 + 337L$.
Hence, we obtain that
$L^2 = 674$.
Thus, it follows that

$$L = \sqrt{674}.$$

Hence, sequence $x_n$ has a finite limit $L$. Therefore, $x_n$ is a convergent sequence.
Example 2. Is the following sequence convergent or divergent?

$$x_n = \frac{n!}{2022^n}.$$

With the ratio test we should check the sequence with n+1 and compare it to n. This results to.

$$x_{n+1} = \frac{(n+1)!}{2022^{n+1}}$$

The ratio tests compare the value of the sequence of n+1 divided by n so we can just multiply by the reciprocal of n

$$\lim_{n\to\infty} = \frac{(n+1)!}{2022^{n+1}} \cdot \frac{2022^n}{n!}$$

We can simply do this more by getting:

$$\lim_{n\to\infty} = \frac{(n+1)!}{2022 \cdot n!}$$

$$\lim_{n\to\infty} = \frac{(n+1)\cdot n}{2022 \cdot n!\;\;!}$$

Then we can write the expression in a way that will allow us to get rid of the factorials on the next step. y we can divide by n!

$$\lim_{n\to\infty} = \frac{n+1}{2022}$$

When $n$ tends to $\infty$ the limit of this expression is greater than 1, which means that given function diverges.

Considered examples show that when we deal with sequences, we need to investigate their convergence and divergence for each particular example separately. Interested reader can find in the literature multiple well-known convergence tests. In this case, we will investigate convergence of sequence using these convergence tests. On the other hand, we will also provide examples of sequence, such that none of the well-known convergence tests work for them and we need to apply non-standard techniques in order to investigate their convergence.

## When Well-Known Convergence Tests Become Inconclusive

For some sequences we don’t need to use any convergence test (such as Ratio test, Root test, Abel’s test, etc.) to determine if the sequence is convergent of divergent. The convergence or divergence of some sequences can be established in a straightforward way without using any convergence tests. On the other hand, there are many examples when one cannot establish convergence or divergence of a sequence and application of a convergence test becomes mandatory.

Example 1. Is the following sequence convergent or divergent?

$$x_n = \frac{n!}{n^n}$$

With the ratio test we should check the sequence with $n + 1$ and compare it to $n$. This results to the following:

$$x_{n+1} = \frac{(n+1)!}{n^{n+1}}$$

The ratio tests compare the value of the sequence of $n + 1$ divided by $n$ so we can just multiply by the reciprocal of $n$

$$\lim_{n\to\infty} = \frac{(n+1)!}{n^{n+1}} \cdot \frac{n^n}{n!}$$

We can simply this more by getting

$$\lim_{n\to\infty} = \frac{(n+1)!}{n \cdot n!}$$

Then we can write the expression in a way that will allow us to get rid of the factorials on the next step.

$$\lim_{n\to\infty} = \frac{(n+1) \cdot n}{n \cdot n!}\ !$$

Finally, we can divide by $n!$

$$\lim_{n\to\infty} = \frac{n+1}{n}$$

When $n$ tends to $\infty$ the limit of this fraction is equal to 1, which means that the ratio test is inconclusive for this particular example.

As we have seen for the above-mentioned example the Ratio test was inconclusive. In some cases, all of the well-known converges tests may become inconclusive. Let us provide such an example.

Example 2. Is the sequence

$$x_n = \frac{1}{n^2 \cdot \cos(n)}$$

convergent or divergent?

First, let us try to find out if this sequence is convergent or divergent by using the ratio test.

We have that

$$x_{n+1} = \frac{1}{(n+1)^2 \cdot \cos(n+1)}$$

Then we divide $x_{n+1}$ by $x_n$.

Hence, we obtain that

$$\frac{(n+1)^2 \cdot \cos(n+1)}{n^2 \cdot \cos(n)}$$

Note that, this expression can be rewritten in the following way

$$\left(1+\frac{1}{n}\right)^2 \cdot \frac{\cos(n+1)}{\cos(n)}.$$

Note that when $n$ tends to $\infty$ then the limit of the last expression is equal to 1, which means the test is inconclusive.

Recall that the ratio test says that if $r < 1$ then the sequence is convergent if $r > 1$ than the sequence is divergent and if $r = 1$ the test is inconclusive. In a similar way, one can check that all other listed tests do not work for this example, which means that none of the tests helps to identify is this a convergent or divergent sequence.

In some cases, other non-standard techniques can be applied either to find the limit of the sequence or to prove that it is divergent. Let us consider the following example.

Example 3 (from Stair-step approach in Mathematics, Sedrakyan).

$$\lim_{n\to\infty} \frac{\sqrt{1+x_(n+1)}-31}{\sqrt{1+x_n}-31} = 0.1$$

Find the limit of the sequence.

Solution: Let $n_0$ be a positive integer, such that for $n = n_0, n_0 + 1$ it holds true

$$0.05 < \frac{\sqrt{1+x_{n+1}}-31}{\sqrt{1+x_n}-31} < 0.15$$

Then if $n > n_0$ then

$$0.05 < \frac{\sqrt{1+x_{n_0+1}}-31}{\sqrt{1+x_{n_0}}-31} < 0.15$$

$$0.05 < \frac{\sqrt{1+x_{n_0+2}}-31}{\sqrt{1+x_{n_0+1}}-31} < 0.15$$

$$0.05 < \frac{\sqrt{1+x_{n+1}}-31}{\sqrt{1+x_n}-31} < 0.15$$

If we multiply all these equations

$$|\sqrt{1+x_{n_0}}-31|\cdot 0.05^{(n-(n_0)+1)} < |\sqrt{1+x_{n+1}}-31| < |\sqrt{1+x_{n_0}}-31|\cdot 0.15^{(n-n_0+1)}$$

This means that

$$\lim_{n\to\infty} |\sqrt{1+x_{n+1}}-31| = 0.$$

From there we find

$$\lim_{n\to\infty} \sqrt{1+x_{n+1}} = 31.$$

Therefore

$$\lim_{n\to\infty} (\sqrt{1+x_{n+1}})^2 = 31^2.$$

$$\lim_{n\to\infty} x_{n+1} = 960.$$

$$\lim_{n\to\infty} x_n = 960.$$

## Acknowledgments

I would like to thank my advisor for the valuable insight provided to me on this topic.

## References

[1] Sedrakyan H., *AIME preparation book*, USA (2022).
[2] Sedrakyan H., Sedrakyan N. *Stair-step approach in Mathematics*, Springer (2018).
[3] Sedrakyan H., *AMC 12 preparation book*, USA (2021).
[4] Sedrakyan H., *AMC 10 preparation book*, USA (2021).
[5] Sedrakyan H., *AMC 8 preparation book*, USA (2021).
[6] “Examples of Sequence Convergence and Divergence - Expii (2022).