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A Study of Chaos in Nonlinear Electronic Circuits and Applications in Game Theory

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ABSTRACT

Chaos is a ubiquitous part of daily life. The weather, fractal landscapes, and the double pendulum all embody the notion of chaos. The general concepts and the key characteristics of chaos are explored and manifested in a nonlinear electronic circuit known as the Chua's circuit. The Chua's Circuit was built and characterized to illustrate chaos behavior in the form of Lorenz attractors. The circuit chaotic behavior was further demonstrated in a numerical model built in MATLAB. More practical chaos applications in secure communication and game theory are discussed. A case study is presented at the end to apply the chaos concept into the model of the strength of poker hands, which displays properties of a chaotic system and resembles Elementary Cellular Automata (ECA) in the chaos application.

Introduction

Chaos is the study of determinism in dynamical systems that are highly sensitive to initial conditions. Weather forecasting, faucet's water droplets, and double pendulums are good examples of chaos. Once thought to describe aperiodic disorder, chaos has evolved into a fundamental branch of mathematics that details everyday life.

Edward Norton Lorenz captured the attention of the world with a seemingly absurd metaphor: the flapping of a butterfly's wings causing a tornado several weeks later. However, the idea of chaos is encapsulated by this statement! The hypersensitivity to initial conditions in a deterministic system creates unpredictability that is not random.

To simulate chaotic behavior in this study, a Chua's circuit was the chosen configuration. In contrast to creating chaos through experimental faucet dripping, which is affected by the slightest vibrations such as footsteps, or through complex simulations of weather, it was determined a Chua's circuit would be the most effective path to examine chaos. In addition, checking the existence of chaos in a Chua's circuit is a relatively simple matter: examination of Lorenz attractors in an oscilloscope is enough evidence to demonstrate path to chaos through bifurcation.

The path to chaos can also be seen in its parallels to game theory. Connections to poker can be established through the ability to model poker as a dynamical system, on top of being chaotic. Elementary Cellular Automata(ECA) is a well-known application and extension of chaos. The comparable properties within ECA and poker point to the modeling of poker in chaos theory through multidimensional analysis.

This paper is organized as follows: Section 2 explores the concepts and theory of chaos as exhibited in nonlinear electronic circuits. Section 3 discusses the experimental setup and the procedure used. Section 4 describes the simulation through the use of MATLAB. Section 5 analyzes the findings of the nonlinear electronic circuits and discusses potential applications in game theory – to model poker as a chaotic system. Section 6 is a summary of the paper.

Theory

There are two topics we must cover to understand the Chua circuit: chaos, and electronic circuits. We will start with chaos.

Chaos

Chaos is illustrated through bifurcation diagrams, which show a route from periodicity into chaos. The simplest way to envision the workings of the bifurcation diagram is through the logistic equation [1]:

$$x_{n+1} = rx_n(1 - x_n)$$

For the purpose of understanding, assume that $r = 2.8$. Thus our equation becomes

$$x_{n+1} = 2.8x_n(1 - x_n)$$

Next, we continuously iterate the function's inputs, given an initial condition of x_0. Specifically, the method of graphical iteration consists of the following steps [1]:

1. Start with the initial condition on the x-axis, and move up vertically to the function $f(x)$.
2. Move horizontally to the $y = x$ line, then move vertically to the function $f(x)$.
3. Repeat steps 1-2.

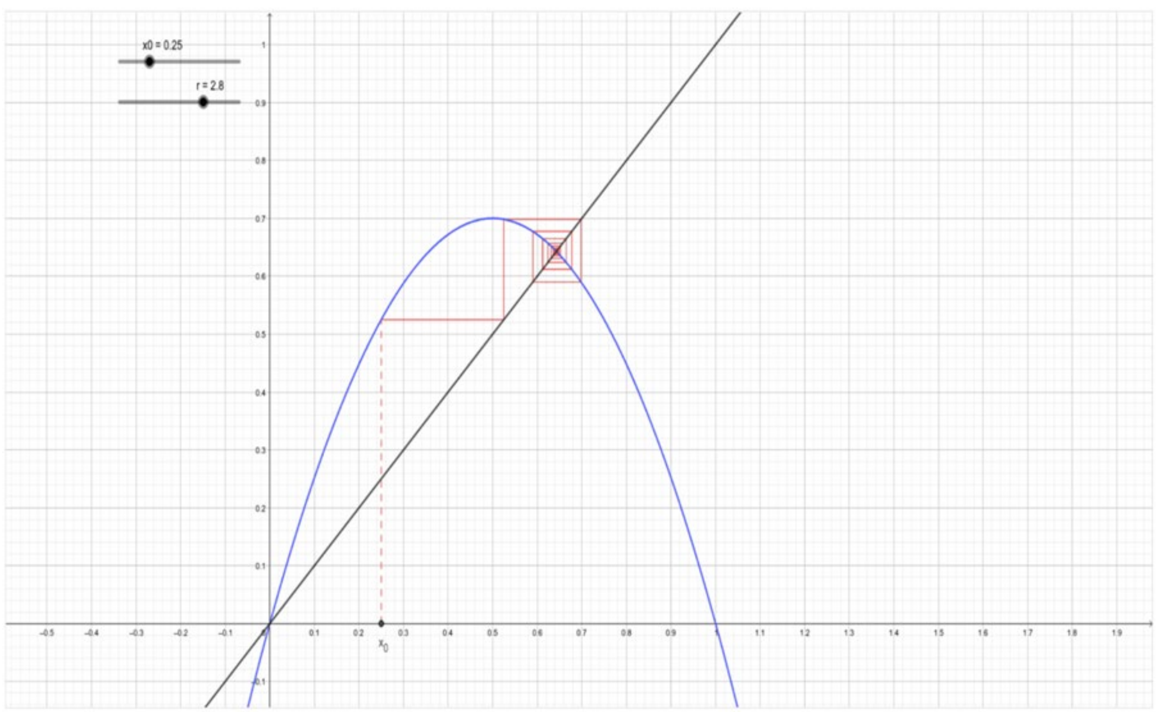

Figure 1. Diagram for successive iterations with $x_0 = 0.25, r = 2.8$ [11]. The values of the successive iterations of x are collectively called the orbit. In other words, we can think of the orbit as a list that represents the "jumping" of the iterations of x from one place to another.

Note how the iterations seem to converge at one point. This point of convergence is approximately at $x = 0.64$. Another word for “point of convergence" is “attracting fixed point”. In this context, we can explain what a bifurcation diagram is: a diagram that allows the reader to quickly determine points of convergence for the logistic equation given different values of both r and x. Effectively, a bifurcation diagram is the logistic map of the equation. In the figure below, chaos, represented by the dark lines, is achieved at approximately $r = 3.6$.

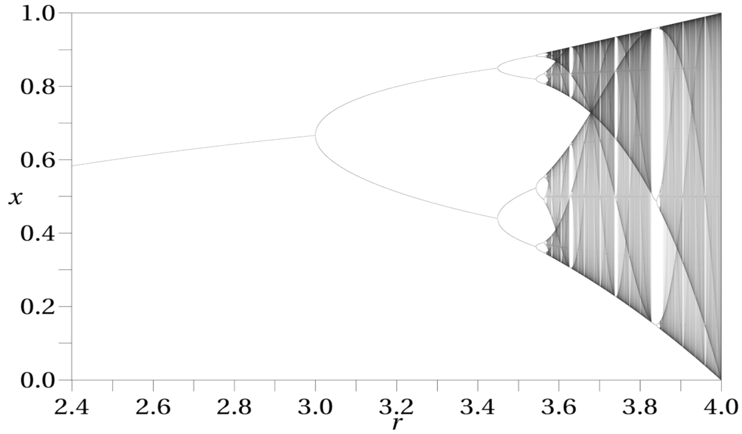

Figure 2. Logistic map and bifurcation diagram for the logistic equation $x_{n+1} = rx_n(1 - x_n)$ (Source: Wikipedia).

To explain the concepts introduced in the previous paragraph with more detail, we will use the term period doubling route to chaos. Given a value of r, the orbit of the logistic equation converges at a single point. This is expressed from $2.4 \leq r \leq 3$. At approximately $r = 3$, the bifurcation diagram displays a split: this represents a periodicity in the orbit between the two values for a fixed r. As r increases, the period of the orbit doubles at specific values of r forever, until dark lines are examined, which represent chaos [1, 2].

A strict definition for a dynamical system to be chaotic is as follows [1]:

1. The dynamical rule is deterministic and has sensitive dependence to initial conditions.
2. The orbits are aperiodic and bounded.

Now let us return our attention to Figure 2. Such bifurcation diagrams extend to all equations. In other words, any equation can be replaced with the logistic equation $x_{n+1} = rx_n(1 - x_n)$, so long as the equation results in orbits that are aperiodic and bounded.

The Feigenbaum constants provide a method of predicting chaos long term by taking advantage of such self-similarity. In particular, Feigenbaum noticed that the ratio of successive intervals between r values that expressed period doubling approached a constant. In other words [1, 3, 4]:

$$\delta_n = \frac{\gamma_{n+1} - \gamma_n}{\gamma_{n+2} - \gamma_{n+1}},$$

where we have

$$\lim_{n \to \infty} \delta_n \approx 4.669.$$

The Feigenbaum constant can be expressed by the distances in Figure 4 [3, 4]:

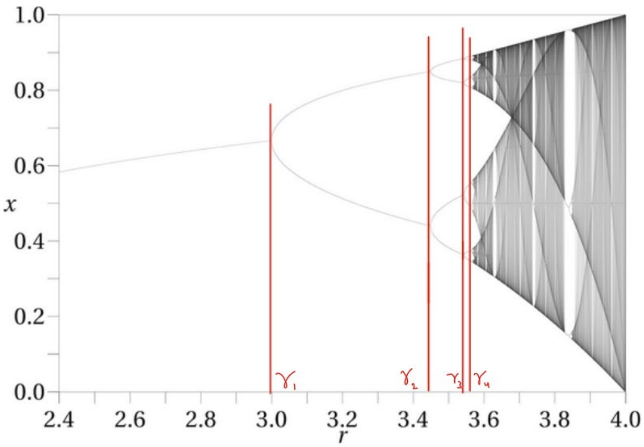

Figure 3. Bifurcation diagram for the logistic equation $x_{n+1} = rx_n(1 - x_n)$, successive values of γ are shown to help visualize the Feigenbaum constant, which essentially states the intervals between the red lines shrinks by approximately a factor of 4.669.

Closely related to bifurcation diagrams are attractors. Attractors are defined as points or regions of space that orbits tend to converge towards. The simplest attractor is a point attractor: in fact, Figure 2 showed an image of a point attractor at $(x, y) = (6.2, 6.2)$. An analogy to help better understand the concept of attractors involves gravitation: points in phase space can be thought to be "gravitationally pulled" towards specific regions of space. One should note, however, that it is not gravity, or any other force that causes the attraction experienced by all points in phase space, but rather, the orbits and paths of points in phase space will converge, purely by deterministic conditions, to fixed regions.

Chua's Circuit

Now, let us direct our attention to the circuitry and workings of the Chua circuit itself. The Chua circuit is a mechanism designed to facilitate chaotic behavior. The diagram of the components of the Chua circuit is shown below.

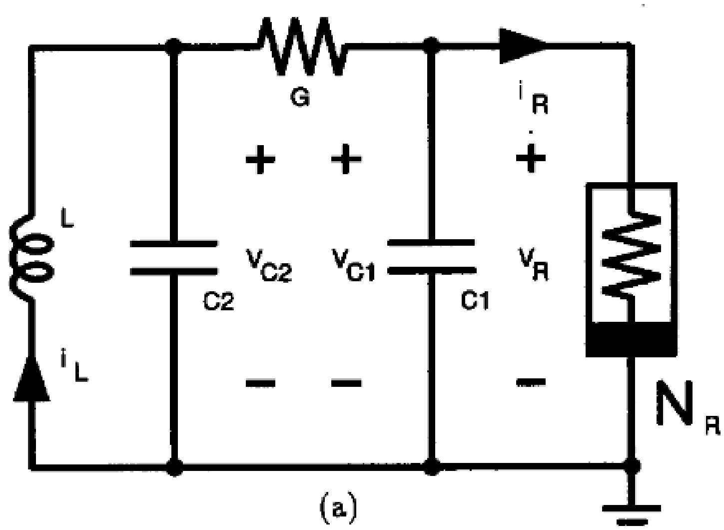

Figure 4. Chua circuit with respective components [5].

In particular, the successive measurement of the voltages across the capacitors, labeled $C1$ and $C2$, when plotted against each other in the xy-plane using the oscilloscope, exhibits chaos in the form of Lorentz attractors. This chaos is made possible through the nonlinear diode, which creates the following voltage and current graph:

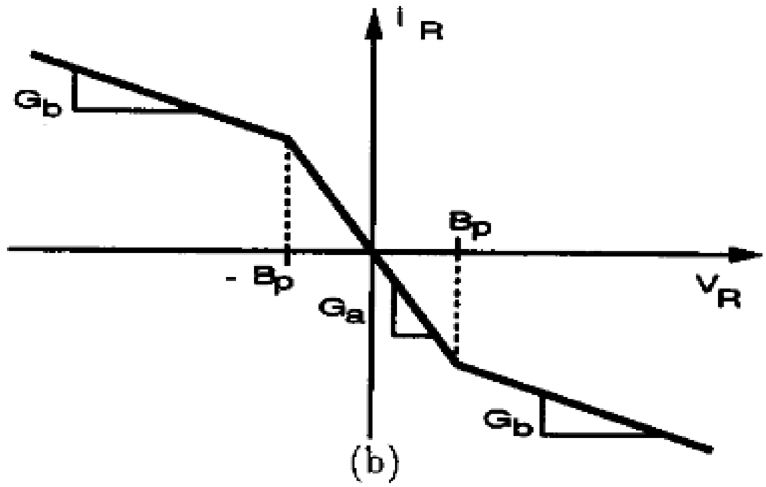

Figure 5. Voltage and current graph [5].

The graph demonstrates how voltage and current, which ideally should behave linearly by Ohm's Law, exhibit nonlinear behavior in the presence of a diode. More explanation on the workings of the diode is in the next section.

An attractor specific to Chua circuits are Lorenz attractors. These attractors are characterized by their relation to Lorenz equations, differential equations used to describe atmospheric convection. These differential equations are examined in Section 4, and their relation to Lorenz attractors occurs from specific conditions: when $\sigma = 10$, $\beta = \frac{8}{3}$, and $\rho = 28$. For the sake of understanding Chua circuits, we need only know that Lorenz attractors act as evidence of chaos when examining the oscilloscope.

Experiment

To carry out the experiment, the following items were used [6]:

- Breadboard
- Op-amp(Operational Amplifier) LMC6482
- Two 22k resistors, two 220 ohm resistors, one 3.3k resistor, one 2.2k resistor
- One 10nF capacitor, one 100 nF capacitor
- One 18 mH inductor
- One 2k potentiometer
- Two 9 V batteries
- Two cable-wire connectors
- Abundant wiring
- Hantek digital oscilloscope

The basic concept of the setup relied on the creation of a diode. A diode is a device that should ideally allow current to travel in only 1 direction. To create a diode, the resistors above are hooked up together, along with the Op-amp LMC6482 that contains the semiconductors necessary for the attraction/repulsion of electrons.

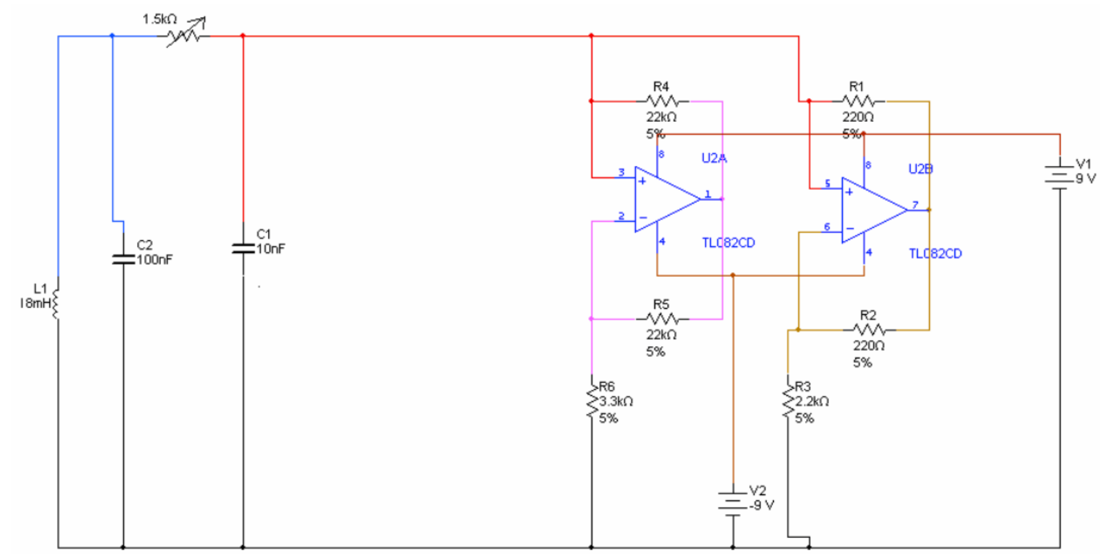

Figure 6. Schematic diagram of the Chua circuit [6].

The operational amplifier will take in two voltages as inputs, and output their difference multiplied by some factor unique to the properties of the device (material, resistance, etc.). It wants the two input voltages to be the same value, and will adjust its voltage outputs over time to resist a difference in voltage. These operational amplifiers work together to simulate the behavior of a diode.

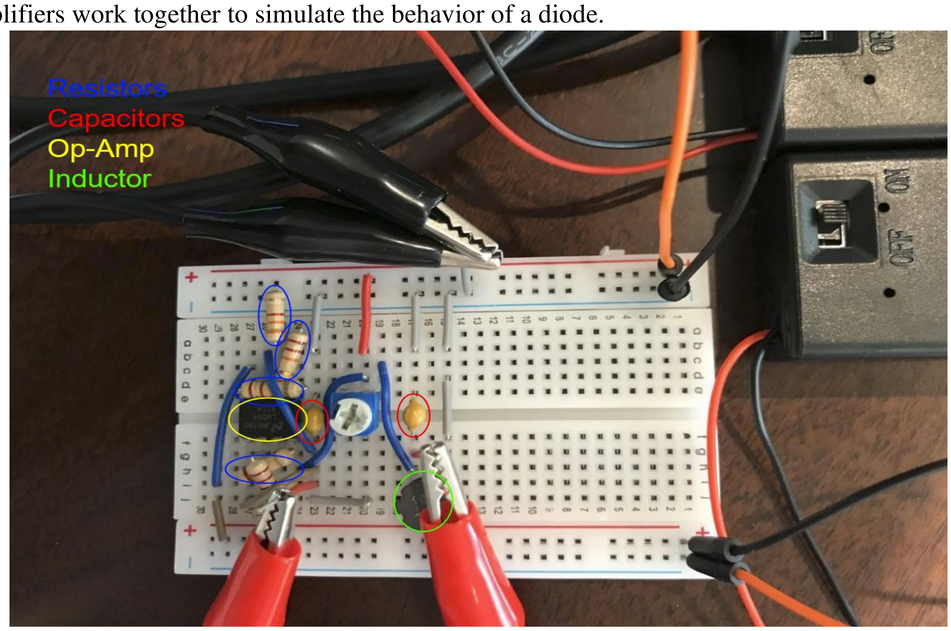

Figure 7. As seen above, blue circles represent the resistors, red circles for capacitors, yellow for the operational amplifier and green for the inductor. The black and red "tongs", as well as the metal on the breadboard serve as wires, while the batteries are in the top right corner.

Following steps outlined in [6], the circuit was built from scratch on a single breadboard. We initially attached the operational amplifiers to the breadboard. After this, everything was built around the operational amplifiers, starting with the resistors in the designated areas, then the capacitors, and finally the inductor. After inserting these key components, wires were carefully cut and inserted to ensure good circuit connectivity and layout readability.

After everything in the breadboard was set, battery powers were applied, and voltages were measured via connection of probes, seen in Figure 6 as the red and black "hooks" connected to parts of wire.
The current is supplied by two batteries, denoted $V1$ and $V2$. These voltages are plotted in Figure 8 for the case when there is periodicity in the orbits of the respective voltages. In this sense, the graph is stable, and the circuit manifests no chaos.

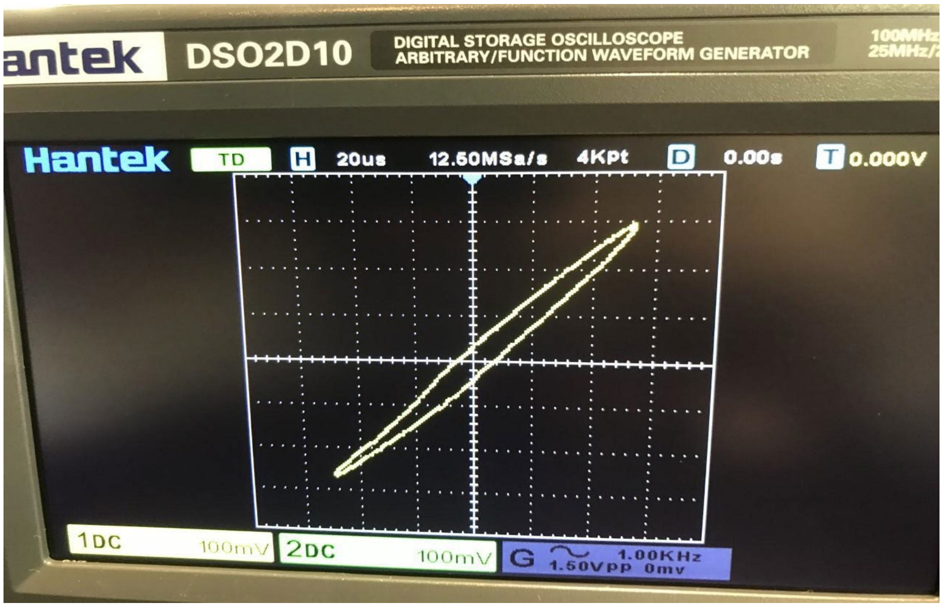

Figure 8. Readings of the oscilloscope for periodicity.

In contrast, Figure 9 shows chaos in the form of Lorenz attractors, after carefully tuning the potentiometer. A more capable oscilloscope would trace out a Lorentz attractor rather than provide the broken images that make up the 3-D image of a Lorenz attractor.

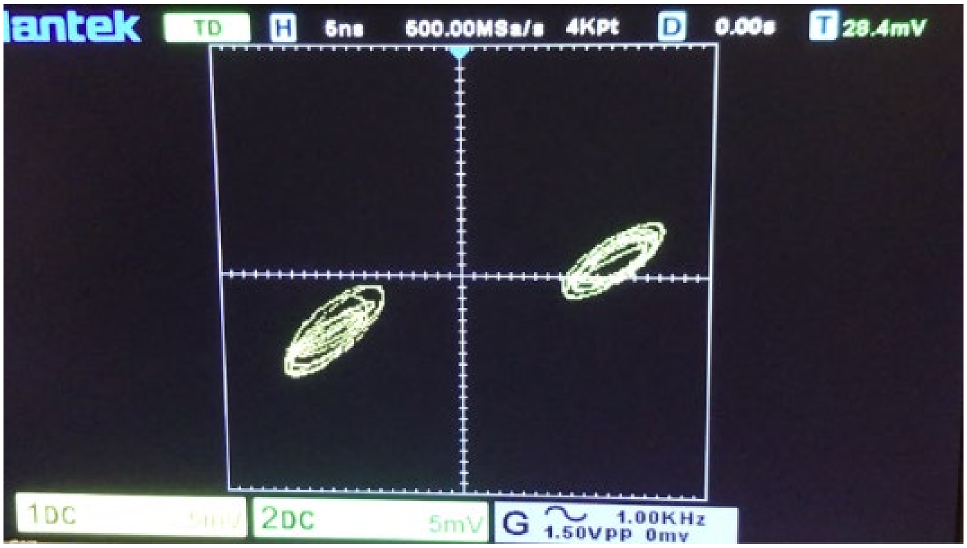

Figure 9. Lorenz attractors present in the plotted xy-plane of voltages. Limitations of the oscilloscope prevent us from seeing more complicated structures.

The resemblance of the shape in Figure 9 can be seen with the butterfly shape so widely known in chaos theory:

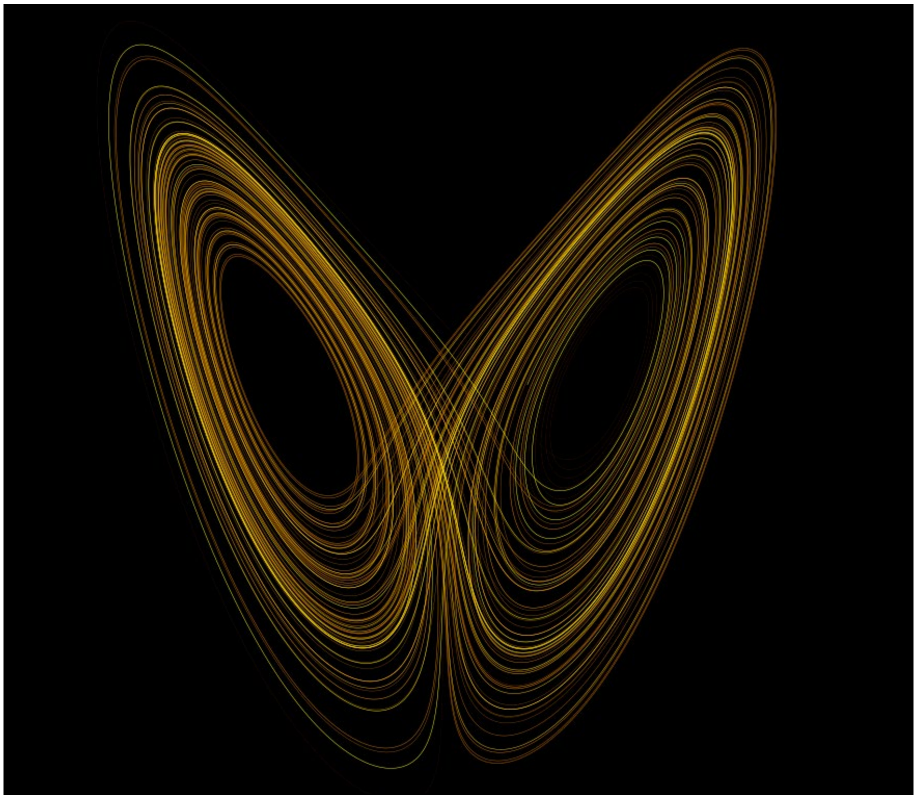

Figure 10. Lorenz attractor, butterfly shape (Source: Wikipedia)

Finally, Figure 11 is a picture of an ohmmeter, with a negative resistance. Chaos is only observed for a certain range of resistances: the negative sign indicates the polarity of the probes of the voltmeter at the time of measurement.

Figure 11. Measure of resistance across resistor.

Simulation

Merely because Chua's is a circuit, equations can be derived using Kirchhoff's theorems. Figure 5 is available as a reference to examine the circuit. Kirchhoff's theorems function on the basis of examining "loops in the circuit". Carefully choosing which loops to use will allow us to obtain differential equations that match the format of the Lorenz equations, seen in Figure 12, which we know to exhibit chaos.

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z$$

Figure 12. Lorenz equations (Source: Wikipedia).

Since, in the Chua circuit, we measure the voltages across the capacitors of the circuit with respect to each other as a determination of the existence of chaos, we ideally should examine loops with the capacitors. These loops are drawn in the figure below:

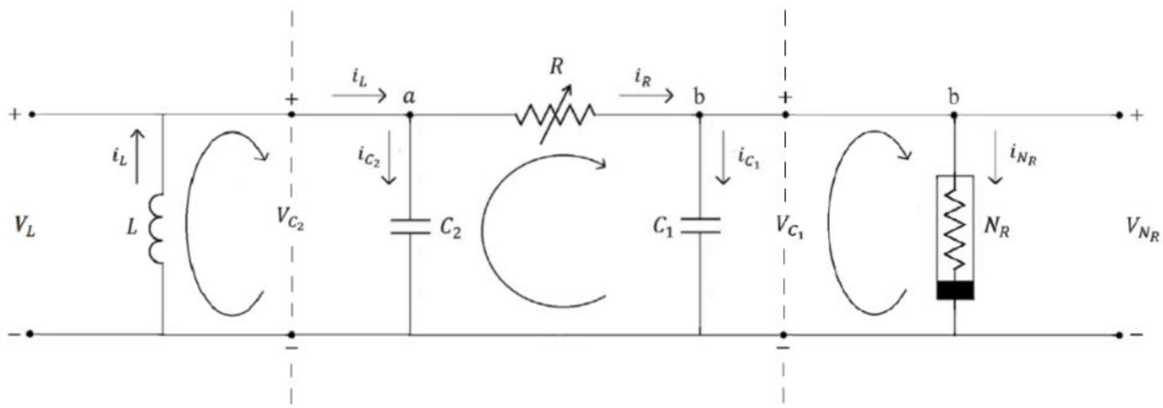

Figure 13. Loops used to derive Chua's equations [7].

By using these three loops, we get three resultant equations, which are seen below:

$$C_1 \frac{dv_{C_1}}{dt} = G(v_{C_2} - v_{C_1}) - g(v_{C_1})$$
$$C_2 \frac{dv_{C_2}}{dt} = G(v_{C_1} - v_{C_2}) - i_L$$
$$L \frac{di_L}{dt} = -v_{V_2}$$

Figure 14. State equations of the Chua circuit [8]. C_1 and C_2 represent the capacitance of the capacitors, v_{C_1} and v_{C_2} represent the voltages across the capacitors, G is the reciprocal of the resistance of the resistor, i_L is the current just below the inductor, and $g(v_{C_1})$ is the change in current with respect to voltage across the diode.

These equations can easily be solved with MATLAB. By entering the differential equations into MATLAB, we are able to see a much cleaner and more apparent appearance of chaos in the form of Lorenz attractors, as depicted in Figure 15.

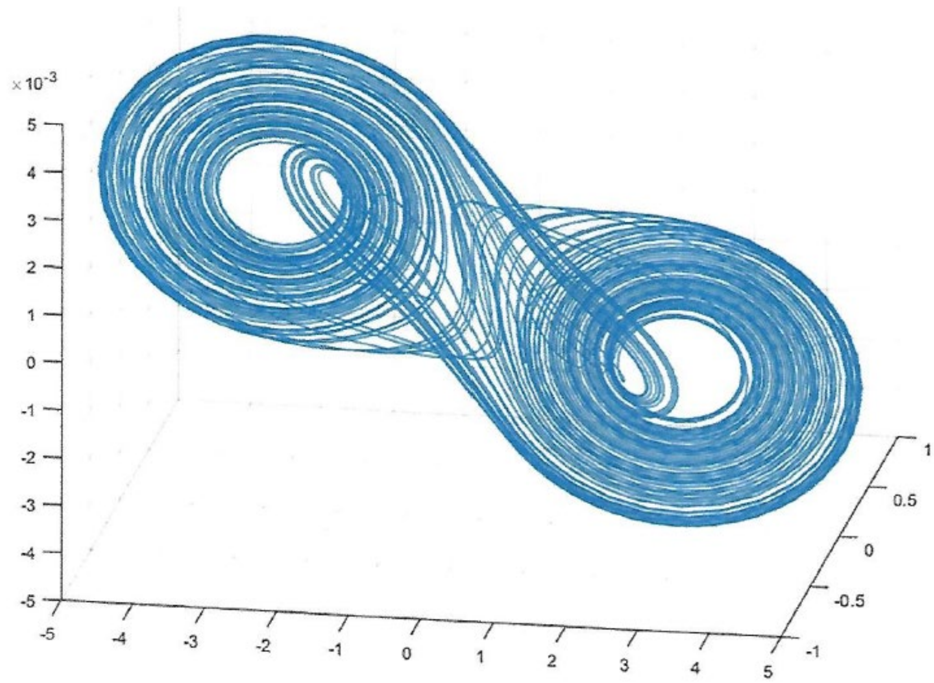

Figure 15. Lorenz attractors from MATLAB simulation of Chua circuit.

Discussion

Many aspects of the experiment could have been improved. For instance, a more capable oscilloscope could provide the means to replicate Lorenz attractors without the need of MATLAB simulation. Additionally, a more tangible improvement would be the calculation of the Feigenbaum constants experimentally. For example, an analysis of the voltages with respect to changes in the potentiometer could have allowed us to calculate period doubling time intervals, which would thus enable us to approximate the Feigenbaum constants.

Chaos has numerous applications in the world, ranging from topics in biology to mathematics, but perhaps the most useful application in modern times is secure communication. The concept is simple enough: when data is encrypted, a key of some sort must be used at both ends of the communication to deny unauthorized access to information. Such a key can be generated through use of a chaotic system, which would render it impossible to guess without prior knowledge of its initial conditions. Although replications of chaos using code are possible, we could alternatively use a Chua circuit to conduct chaotic behavior to encrypt messages! Essentially, secure communication is possible even with a given rule set available to the public because chaos' sensitivity to initial conditions deems encrypted data impossible to read.

A possible future extension and application of chaos theory lies in poker dealing [9]. Poker is a complex game that could exhibit chaos. While it is true that elements of poker, mostly the human aspect of the game, cannot be altered or accounted for by pure mathematics, it is noteworthy to mention that the strength of hands in poker exhibit chaotic behavior.

To start with, the "relative strength" of hands is a quite subjective matter, as the strength of hands is never fixed by hierarchy. In some situations, even weak hands can prevail in terms of strength as the cards are continued to be dealt. In addition, numbering such rankings is difficult, and points in space close to each other arguably function the same. Thus, the strength of hands will be constructed through winning percentages, i.e., the probability of winning in a given situation with a given hand.

Chaos is evident in the correlation between its properties and rules of poker. In particular, the definition of chaos is where a dynamical system has certain properties (see Sec. 2.1). It is interesting to note that poker obeys these properties. Poker can be viewed as a dynamical system because each outcome, each possibility could be expressed as a point in space, where the story of the hand changes because of a time dependence. A more elaborate description is as follows. The relative strength of each person's hand can be used as an axis in phase space, which could very well lead to higher dimensional space. As such, as time progresses, the specific point in this space can shift as a new communal card is played and different cards are taken into account. Additionally, poker is deterministic: once the cards are shuffled, each card to follow is set and unchangeable. Everything, from the initial hands of players to the communal cards have set values from the beginning, very similar to how the logistic equation has a fixed order once given an initial condition. Despite this, poker has sensitive dependence to initial conditions, as a change in the order of cards, ever so slightly, could alter the strength of poker hands.

Another angle to consider when viewing the chaotic nature of poker is poker's relations to Elementary Cellular Automata(ECA). ECA is the study of one-dimensional states that are either 0 or 1, where each subsequent "stage" is dependent on previous states [1]. The resemblance to poker is in its bounding and dependence to neighboring positions. ECA is restricted to its set amount of initial conditions that progress to chaos in some cases. In a similar manner, poker is constrained to 52 cards, and the strength of hands are determined by the initial order. ECA is determined by cells adjacent, just as poker hands are determined by neighboring cards: many poker games give players multiple cards as their own. The key difference between ECA and poker is through the progression of states with respect to time. ECA is able to continue forever, while poker is only allowed to continue for a limited amount of rounds. However, since poker follows the "rules" of chaos, it can still classify as being chaotic.

Conclusion

A construction and simulation of the Chua circuit was conducted, and it was confirmed that chaos existed in both. Upon varying the parameter of frequency within the diode, chaos was observed in the LRD circuit because of the Lorenz attractors observed when plotting the voltages across the capacitors against each other. It was the period doubling that was largely responsible for this: for some frequencies, the observed pattern was a convergence to a point, others to a line, but it was chaos that saw the variance of the Lorenz attractors. The similarity of chaos within the circuit and concepts of ECA indicates a connection between chaos and poker. The valid characterization of poker as a dynamical, chaotic system, along with the bounding and dependence to neighboring positions of ECA that mimic parts of poker together show chaos' connection to poker.

Acknowledgments

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References

1. Feldman, David, Chaos and Fractals: An Elementary Introduction, Oxford, Oxford University Press, 2012
2. Propp, S. Period Doubling Route to Chaos in a RLD Circuit.
3. Hanias, M. P., Avgerinos, Z., & Tombras, G. S. (2009). Period doubling, Feigenbaum constant and time series prediction in an experimental chaotic RLD circuit. *Chaos, Solitons & Fractals*, *40*(3), 1050-1059. https://doi.org/10.1016/j.chaos.2007.08.061

4. Alam, J., Anwar, M. S., & Latif, U. A. (2015). Non-linear dynamics with an RL-Diode Circuit. *LUMS School of Science and Engineering, Version*, *1*.
5. Chua, L. O., & Huynh, L. T. (1993). Bifurcation analysis of Chua's circuit. In *MIDWEST SYMPOSIUM ON CIRCUITS AND SYSTEMS* (Vol. 35, pp. 746-746). PROCEEDINGS PUBLISHED BY VARIOUS PUBLISHERS. https://doi.org/10.1109/mwscas.1992.271217
6. Gandhi, G., Muthuswamy, B., & Roska, T. (2007). Chua's circuit for high school students. *International Journal of Bifurcation and Chaos*, *12*.
7. Limphodaen, N., & Chansangiam, P. (2020). Mathematical analysis for classical Chua's circuit with two nonlinear resistors. *Songklanakarin J. Sci. Technol*, *42*, 678-687.
8. Tipler, P. A., & Mosca, G. (2007). *Physics for scientists and engineers*. Macmillan.
9. Jóźwiak, I. J., Mariański, A., Switana, J., & Szczepanik, M. (2020). Applying chaos theory to risk management in a game of chance. *Zeszyty Naukowe. Organizacja i Zarządzanie/Politechnika Śląska*.
10. http://www.chuacircuits.com/MATLABsim.php
11. https://www.geogebra.org/m/cjvqjcbh