# Investigating the Flow Rate of Granular Material in a Dynamic System: How Does a Constant Change in The Silo's Angle of Inclination Impact the Granular Flow Rate?

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ABSTRACT

The aim of this research is to investigate the effective method of predicting granular flow rate in a dynamic system, specifically a system undergoing vertical periodical fluc- tuation, simplified into uniform circular motion in this investigation. The preliminary investigation recognises the explanation from Beverloo's relation in literature and con- firmed its effectiveness in predicting the flow rate of granular material through a static silo. Hence, a mathematical hypothesis regarding the research question is established successfully, synthesizing the hypothesis of projection of effective diameter and vertical acceleration. In experiment, the hypothesis of the projection of effective diameter of the flow region is verified experimentally; whereas various implications arise when applying the conventional force analysis into the Beverloo's relation. Although the current evidence is deficient in supporting me proposing the new law, this investigation systematically discuss the inter- connection among the variables in Beverloo's relation, and the current implication revealed shed a light on fur- ther research.
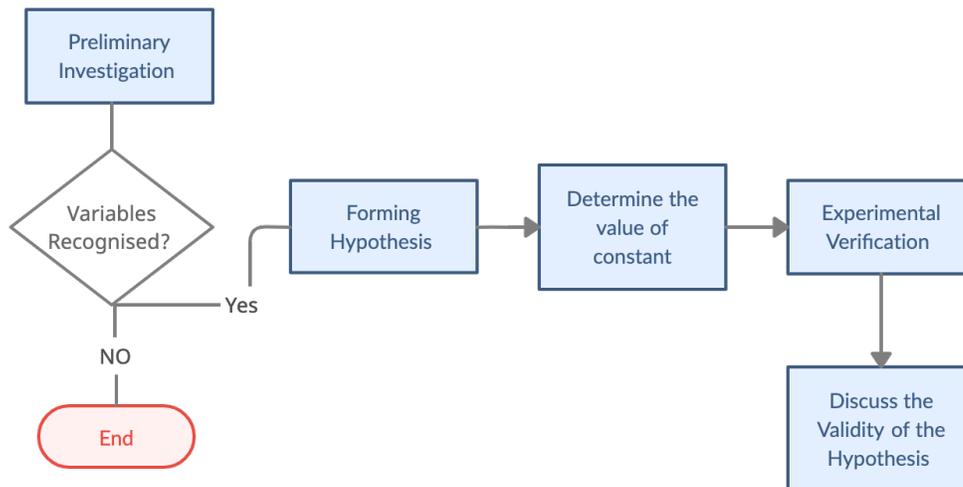
Flow Chart 1. The investigation process of this extended essay (Author's Own

Introduction

Background Information

Granular materials are the second-most manipulated material after water in industry (Richard et al), making up most of the fundamental objects in the world, from restaurant hourglasses to industrial silos. Therefore, especially in the construction and manufacturing industry, con- trolling its flow rate has been a significant issue regarding the safe and efficient storage, trans- portation and packing of this dominant material over centuries. (Mankoc et al, 1) One of the most common methods to accelerate the granular discharge rate to improve the efficiency of transportation is vertical angular vibration in factories. Apart from industrial application, the research of granular flow rate is closely connected to the stability assessment of ground surface collapse in natural disasters and soil loss. (Mohensen,1)

Figure 1. Industrial granular material silos (AMMAG)

However, complications arise constantly due to the material's non-linearity, suggesting its condition is highly dependent on experimental conditions from a microscopic perspec- tive(Serrano et al, 139), mostly due to its sensitivity to pressure, which constantly challenges sci- entists to mathematically understand the dynamics of the bulk materials (Sheldon and Durian, 579). The broad research prospect and wide applicability in geology, agriculture and raw ma- terial industries aroused my interest, shedding light on the possibility of directly contributing to this fascinating area of research.

Research Question

How does a constant change in the silo´s angle of inclination impact the granular flow rate?

Preliminary Investigation

In this section, I will mathematically and experimentally review my primary theory: Beverloo's relation, in predict- ing the granular flow rate through a horizontal static orifice. This preliminary investigation will assist me to better

understand the interaction among the essential variables and parameters in the granular flow system and to there-fore mathematically manipulate the relation in my further investigation of the dynamic system.

Literary Investigation of the Beverloo's Relation

Developed in 1960(Beverloo, 260), the Beverloo's equation is derived from abundant experi- mental data and proven to be effective in describing various patterns in the last 60 years, i.e. intermittent flow, funnel flow, etc(Mankoc et al, 2). Mathematically, the relation is expressed as (Beverloo, 269):

$$W = C\rho bg^{1/2}(D0 - kdp)^{5/2} \qquad (2.1.1)$$

where the granular discharge rate is defined as W, with unit kgs^-1. $D_0(m)$ is the diameter of the orifice, $d_p(m)$ is the mean diameter of the grain, $\rho_b(kgm^{-3})$ is the bulk density of the material, defined as the weight of the grain in a specific volume. C, g, k are parameters, with C being the empirical discharge coefficient related to the friction coefficient(μ) on the circum- ference near the orifice, g being the gravitational acceleration($9.806ms^{-2}$), and k being the empirical shape coefficient, related to the dynamic distribution of the material inside the silo.

From Figure 2, the inner pile of granular material is divided into three regions: the flow Area, which is dynamic throughout the discharge process; the static area, which is temporarily static and constantly going through formation and break-up(Sheldon and Durian, 2); the stable re- gion, in which the material never discharges. Past literature suggests that the distribution of the three regions is directly linked to the grain diameter(Guo and Zhou, 7), which rationalises the role of as a coefficient modifying the proportionality between d_p and the volume of the static area. Moreover, it can be further interpreted that the effective diameter affecting the flow is the diameter of the flow area shaded in grey. This is however not always consistent with the actual diameter if there are any variations in the position of the system, further substantiating the significance of the modifying constant.
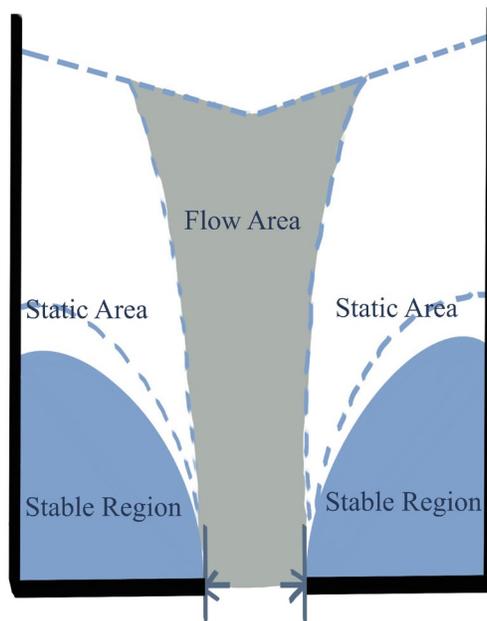

Figure 2. Illustration of a cross section of a static granular flow system, not to scale(Author's Own)

Additionally, the factor $(D_0 - kd_p)$ indicates an effective orifice size, that only when $D_0 - kd_p > 0$ there will be a flow present. As experiment suggests that $k \in (1, 2)$, it's only possible for grains to drop when $D_0 > 2d_p$(Serrano, 139). To guarantee producible ad constant flow, Beverloo further restricts that the relationship is

valid only when $D_0 > 6d_p$. This enables me to consider that within the limit $\frac{D0}{d} \gg 1$, the following relation is valid:

$$W \propto C\rho_b g^{1/2} \ D^{5/2} \tag{2.1.2}$$

(Beverloo,267)

Equation (2.1.2) appears similar to the form of Hagen's Principle(Serrano and Ruiz-Chavarr´ıa 139), which is written as $W = C\rho_b g^{\frac{1}{2}} D^2$. According to my derivation, it can also be assumed that the equate sign holds true when the particle size is negligible, and the equation can there- fore be applied in predicting the flow rate of powder and sands. Hence, the validity of the assumption (2.1.2) is cross-substantiated.

Experimental Investigation of the Beverloo's Equation

Figure 3. the lids of the silo used in the experiment, with orifices made by laser cut and sizes designed by Adobe CAD(Author's Own)

In this investigation, I set the diameter of the orifice D_0 as my independent variable, and the dependent variable is the discharge rate. By maintaining a static horizontal orifice and using the same material consistently, I hypothesise that, according to equation(2.1.2), a directly proportional relationship should be obtained between the discharge rate W and $D^{5/2}$

To prepare the experiment, the independent variable is controlled by making multiple wooden lids through laser-cut to minimise the random error in measurement, shown in Figure 3. The unit of the diameter D_0 is taken in *mm* instead of *m* in the original equation to better demon- strate the change in independent variable. The decrease in scale of the horizontal axis will not impact on the result, as the experiment aims to examine the correlation.

Figure 4 presents the set-up of the experiment. The silo is firstly filled with the sands and the weight of the sands together with the silo is measured. Then, the silo is vertically fixed on the retort stand up-side down, and the orifice is blocked manually. The sands are released from the silo once start timing. After a considerable period of time, the orifice is re-blocked, and the timer is stopped immediately. Measure the final mass of the silo with sands left, and the difference between the initial mass and the final mass taken of the system is the mass of the sands discharged.
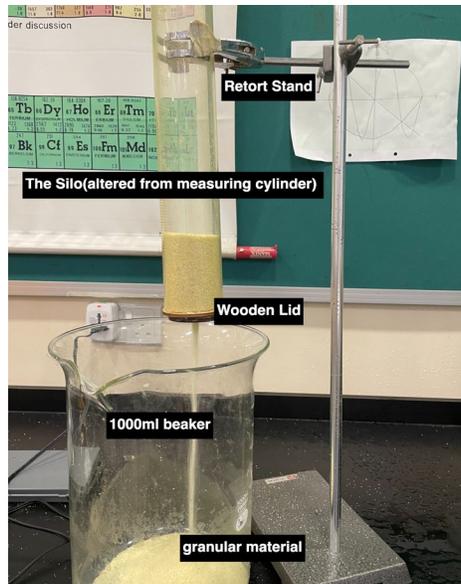
Figure 4. Experiment set-up for the preliminary investigation(Author's Own)

Notice that the time taken and the packing height of the granular material aren't necessary to be controlled, due to the fact that the variables do not appear in the relation, nor does any additional theory indicate that packing height is relevant in the experiment. This can be explained by the fact that the formation of the arch is affected by the pressure distribution around the silo, both independent of the height of the grain.

Table 1. Raw data table for the investigation of the granular discharge rate through a static horizontal orifice based on Beverloo's Relation, the uncertainty in the measurement of mass is 0.01g, the uncertainty in the measurement of time is 0.01s

Diameter/D_0 (mm)	Run 1			Run 2			Run 3		
	Initial mass(g)	final mass(g)	time(s)	Initial mass(g)	final mass(g)	time	Initial mass(g)	final mass(g)	time(s)
2									
3									
4	891.12	810.68	64.85	810.68	725.25	66.70	725.25	642.79	72.97
7	885.76	611.96	36.64	611.96	386.70	29.69	386.70	148.80	33.16
8	874.61	690.98	21.22	690.98	546.13	14.87	546.13	150.40	41.73
10	853.92	595.72	14.14	595.72	146.49	24.64	875.11	193.57	38.93
11	871.81	556.39	12.31	556.39	143.47	15.87	866.31	152.86	26.99
12	780.74	140.54	19.62	808.13	132.91	24.31	853.54	145.81	21.23
14	817.37	140.63	15.74	849.97	142.73	15.67	836.45	136.70	15.93

Note that in the shaded region of the table, i.e. when $D_0 = 2mm$ and $D_0 = 3mm$, intermittent granular flow is observed, suggesting that the valid range of D_0 is when $D_0 \geq 4mm$ (as the range of kd_p is tested to be $kd_p \in [3, 4)$). Therefore, these two shaded sets of data collected are invalid.
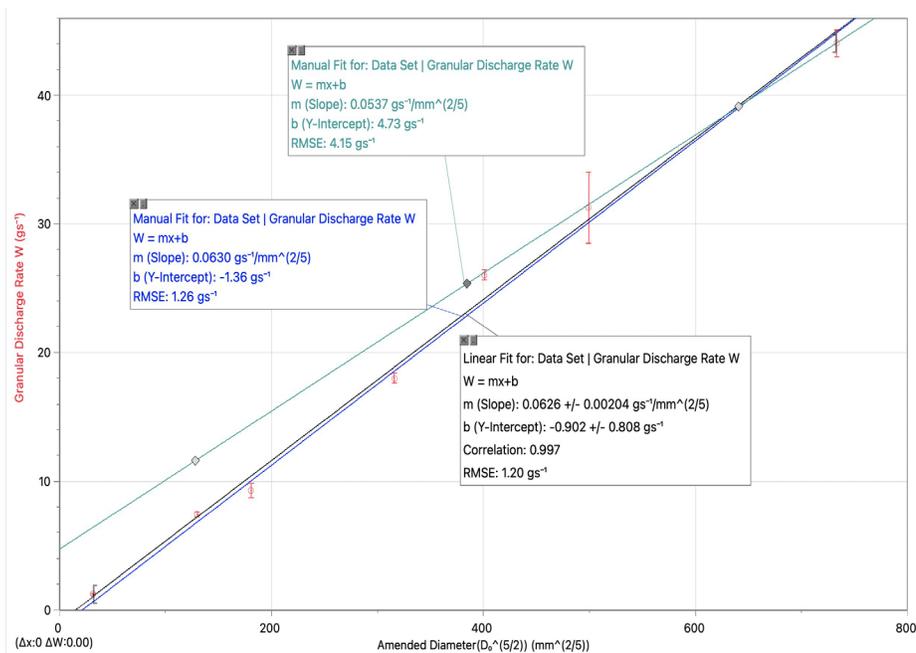
Table 2. Sample Calculation and Uncertainty propagation

Sample Calculation	Uncertainty Propagation

example taken in the first trail when $D_0 = 4mm$.
Sample discharge rate:
$(891.12 - 810.68) \div 64.85 = 1.24gs^{-1}$ (2d.p.)
Sample average discharge rate over three trails when $D_0 = 4mm$
$(1.24 + 1.28 + 1.13) \div 3 = 1.22gs^{-1}$(2s.p.)

Percentage uncertainty in measurement:
$(0.02 \div (891.12 - 810.68)) \times 100\% + 0.01 \div 64.85 \times 100\% = 0.04\%$(1s.g.)
Uncertainty over three trails (1d.p.)
$((1.28 - 1.13) \div 2) \div 1.22 \times 100\% = 6.2\%$
Compare the percentage uncertainty:
$6.2\% > 0.04\%$ the larger percentage uncer- tainty is taken
Absolute Uncertainty
$6.2\% \times 1.22 = 0.08gs^{-1}$(2d.p.)

Table 3. Processed Data Table for experimental investigation of the granular dis- charge rate through a horizontal static orifice, the unit of percentage uncertainty is % (1d.p.)and the unit of absolute uncertainty is gs^{-1} (2d.p.)

D^2 (mm^2)	Average discharge rate (gs^{-1})(2d.p.)	Percentage uncertainty	Absolute Uncertainty
32	1.22	6.2	0.08
130	7.41	2.8	0.21
181	9.29	5.9	0.54
316	18.00	2.1	0.38
401	26.03	1.6	0.41
499	31.25	8.9	2.78
733	44.02	2.4	1.07

Graph 1. The linearised and annotated graph of data collected in the investigation of granular discharge rate through a static arch. The black line is the line of best fit, the blue line is the maximum gradient and the green line is the minimum gradient

Discussion of the Result of the Preliminary Investigation

The first aim of the preliminary investigation is to test Beverloo's law 's accuracy under my lab condition. This is reflected by the strength of the linear relationship between the two variables: W and $D^{5/2}$. From the graph, I obtained the equation of the line of best fit is $y = 0.0626x − 0.902$(Correlation Coefficient: r=0.997)(RMSE=1.2). The Root-Mean-Squared Error (RMSE)and correlation coefficient correlation (R) are in a very good range, i.e. close to 1, quantitatively showing the linear relation among the set of data points between the amended diameter and the discharge rate is strong from multiple perspective, which fits my hypothesis. As for error analysis, the maximum and minimum gradient for the line of best fit is drawn from error bar, which is respectively:

$$y = 0.0630x − 1.36(3s.f.)$$
$$y = 0.0537x + 4.73(3s.f.)$$

The percentage uncertainty in the gradient: $(0.0630 − 0.0537) ÷ 0.0626 × 100\% = 14.8\%$ a systematic error can be interpreted from the presence of y intercepts in the three functions. However, considering the range of y value is relatively large compared to the value of intercept with the sign varying between the minimum and maximum gradient, the systematic error in the experiment therefore has a marginal impact on the effectiveness of Beverloo's law. However, a considerable random error is present in my experimental investigation, suggesting potential amendment could be done to improve the accuracy of this experiment in further investigation. One of the significant sources of random error is identified as the measurement of time, as evidenced by the variation in the mass taken. To improve the accuracy of the measurement, a wooden slip is added to the lid in the next section to facilitate a more accurate measurement in time.

The second aim of this investigation is to achieve a better understanding of the variables in the Bev-

erloo's relation that could potentially impact the prediction when the system undergoes circular motion. From the theoretical investigation, it's indicated in the past literature that the condition around the orifice is crucial to the formation of the static and stable arch and therefore the discharge rate, and the change in inclined angle could impact such condition. Three variables interpreted in section 2.1 describe the condition around the orifice: the constant C, the potential change in friction; k, the shape of the arch around the orifice; both impact the projected horizontal area of the orifice(Sheldon and Durian, 582), with g describing the vertical motion. The good linear relationship in experimental investigation further confirms that C and k are independent of the value of D_0, and remain stable in the static and horizontal condition. I will therefore conclude the preliminary investigation and set my new aim in the next section to further investigate the variation in C, k, g and D_0.

Beverloo's Equation in Dynamic System

In this section, I will investigate variation in granular discharge rate in a dynamic system, specifically the change in the values recognised in the Beverloo's relation. The aim of this section is to formulate a mathematical hypothesis which is able to predict the granular discharge rate undergoes uniform circular motion.

Figure 5. Industrial Aerators: an industrial method generating granular flow undergoing partial circular motion(AKO UK Pinch Valves)

Theoretical Hypothesis

Based on the previous investigation, it can be inferred that the discharge rate of granular ma- terial with a constant change in silo's angle inclination differs from the vertical static system by two factors:

1). The inclined angle interrupted the formation of the static and dynamic arches, hence the effective flow diameter illustrated in Figure 2. The value describing this property are the con- stant C and k (Hunt et al.)

2). The rate of change in inclined angle attributes to the vertical acceleration of the grains (Wassgren et al.).

Nonetheless, important assumptions have to be noted before starting my mathematical investigation. Firstly, the flow rate is assumed to be consistent by using Beverloo's equation, while the flow rate in reality is unsteady, evidenced by the phenomenon of ticking hourglass(Maloy and Hansen, 1363). Secondly, the density distribution of the grain inside the silo is assumed con- stant; while a slight variation might have occurred due to the vibration. (Evesque and Meftah, 1801). This change in density distribution could potentially cause fluctuation in instantaneous flow rate. Having the two assumptions will not impact the investigation of average flow rate

significantly and hence the effectiveness of the mathematical hypothesis. However, the extent of impact of the two assumptions is still to be determined by experiment, which requires more precise measuring experimental apparatus.

Before investigating the dynamic system, I will first analyse the impact of the change in angle on the effective diameter of the free flow area. This in the preliminary investigation is reflected by the change in constant C and k. I will therefore use $\frac{W}{W_0}$ to represent the combined variation in the constant C and k, with W representing the discharge rate with inclined orifice and W_0 the discharge rate with horizontal orifice. In the preliminary investigation, the flow area is always vertical, therefore its effective diameter is D_0. As a inclined angle is imposed in the system, the effective diameter of the flow area is therefore hypothesised to be the projected length of D_0 on a horizontal line. As Figure 6 shows, define the angle of inclination as θ, the length of projection of D_0 on the horizontal plane is $D_0 \cos \theta$. This hypothesis can be therefore written as:

$$\frac{W}{W_0} = l\cos \theta + n \tag{3.1.1}$$

with the value of l and n determined by experiment. Rearrange the equation to isolate W:

$$W = lW_0 \cos \theta + n \tag{3.1.2}$$

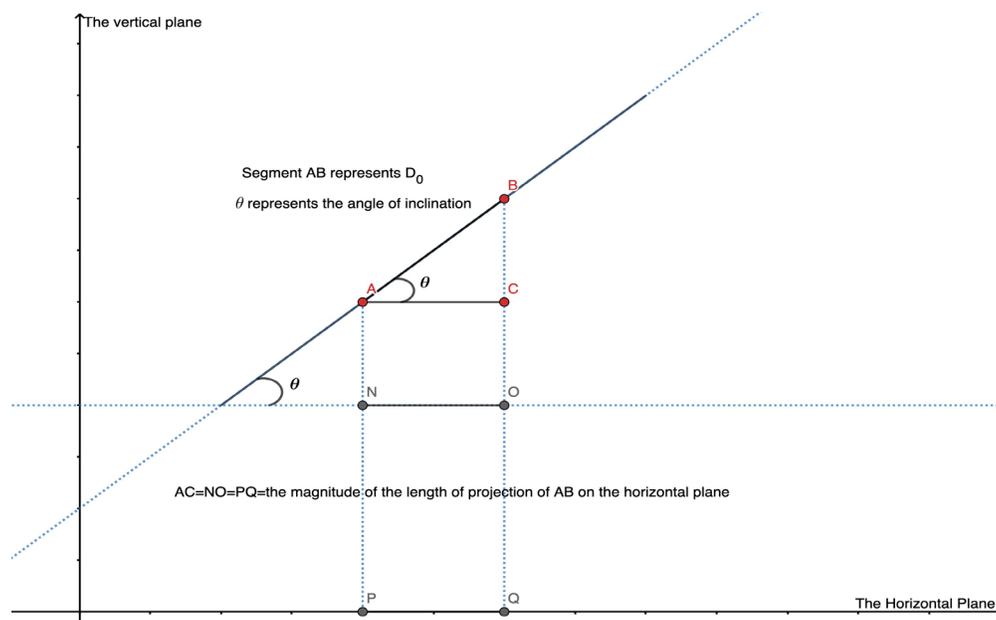This hypothesis concludes the prediction of the granular flow rate with static inclined orifice.

Figure 6. Illustration of the length of projection of D_0

Next, I'll analyse factor (2): the vertical acceleration to replace g 2 in Beverloo's equation under static condition. As the angular velocity in uniform circular motion is constant, θ, the angle of inclination can be expressed by time t and the angular velocity ω as:

$$\theta = \omega t \tag{3.1.3}$$

Referring to the equation of centripetal acceleration a_c, with r being the radius of the circular motion:

$$a_c = \omega^2 r \tag{3.1.4}$$

As a component of centripetal acceleration is attributed to the original vertical gravitational acceleration. As the component's direction varies with the angle of inclination, the total accel- eration can be expressed as following with domain restricted:

For $\omega t \in [2n\pi, \frac{\pi}{2} + 2n\pi], n \in Z^+$

$$a_y = g - \sin\theta a_c = g - \sin(\omega t)\omega^2 r, \tag{3.1.5}$$

For $\omega t \in [2n\pi + \frac{\pi}{2}, 2\pi(n+1)], n \in Z^+$

$$a_y = g - \cos\theta a_c = g - \cos(\omega t)\omega^2 r \tag{3.1.6}$$

Therefore, extract the original acceleration $g^{\frac{1}{2}}$ from W_0:

$$\frac{W_0}{g^{\frac{1}{2}}} = C\rho_b(D_0 - kd_b)^{\frac{5}{2}} \tag{3.1.7}$$

As W_0 is assumed to be constant and hence can represent the instantaneous discharge rate, therefore W can be also written in an derivative function against time as :

$$W = \frac{dm}{dt} \tag{3.1.8}$$

This enables me to combine the result listed above (i.e. (3.1.1), (3.1.6), (3.1.7) and (3.1.8)) and formulate the mathematical equation predicting instantaneous discharge rate under uniform circular motion, only through ω(the angular velocity), t(the time taken from the start), r(the radius of the circular motion), and other physics parameters defined above . The instantaneous discharge rate is written as following:

For $\omega t \in [2n\pi, \frac{\pi}{2} + 2n\pi], n \in Z^+$

$$W = \frac{dm}{dt} = (g - \sin(\omega t)\omega^2 r)^{\frac{1}{2}}(\frac{W_0}{g^{\frac{1}{2}}})n + l(g - \sin(\omega t)\omega^2 r)^{\frac{1}{2}}(\frac{W_0}{g^{\frac{1}{2}}})\cos(\omega t) \tag{3.1.9}$$

For $\omega t \in [2n\pi + \frac{\pi}{2}, 2\pi(n+1)], n \in Z^+$

$$W = \frac{dm}{dt} = (g - \cos(\omega t)\omega^2 r)^{\frac{1}{2}}(\frac{W_0}{g^{\frac{1}{2}}})n + l(g - \cos(\omega t)\omega^2 r)^{\frac{1}{2}}(\frac{W_0}{g^{\frac{1}{2}}})\cos(\omega t) \tag{3.1.10}$$

Recognising that the hypothesis needs to be verified separately, two experiments will be con- ducted, with one confirming the hypothesis of the effective diameter as the length of projection, and its results will be applied to the final experiment of granular flow in the dynamic system.

Experimental Investigation of Effective Diameter

In section 3.1, the relationship between the angle of inclination and the combined change in coefficients C and k is hypothesised as

$$\frac{W}{W_0} = l\cos\theta + n \qquad (3.2.1)$$

This experimental investigation therefore not only aims to confirm the linear relationship be- tween $\cos\theta$ and $\frac{W}{W_0}$, but also find the magnitude of l and n to a specific D_0. Therefore, independent variable of the investigation is the angle of inclination θ and the dependent variable will be the ratio of the granular discharge rate with specific angle of inclination to the granular discharge rate through a horizontal silo, i.e. $\frac{W}{W_0}$.

In the experiment, the orifice diameter of 4mm is chosen whereas 8mm and 10mm are also tested for consistency of the accuracy of the hypothesis in all diameters (3.2.1), with the in- clined angle selected ranged from 10° to 90°. The range is restricted as trail run suggests that no constant flow is observed when the angle is above 90°. If the hypothesis(3.2.1) holds true, a linear graph should be created by plotting $\frac{W}{W_0}$ against $\cos\theta$.
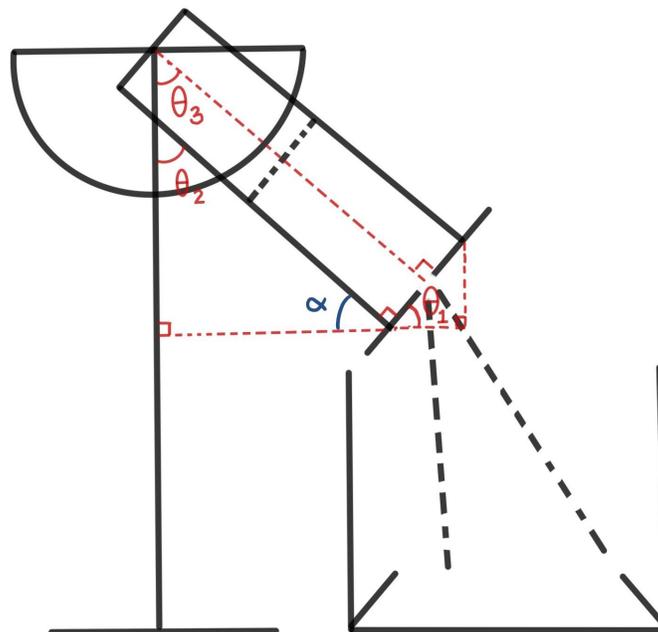

Figure 7. Illustration of the experiment set-up. The apparatus includes a protractor, a retort stand, a silo, a lid with designed diameter, a big beaker collecting the discharged grain(Author's Own)

Figure 7 presents the experiment set-up. The measurement of flow rate, the dependent variable re- mains consistent with the method of preliminary investigation, calculated as the mass of grain(taken in g) discharged over a unit of time (taken in s).The independent variable, the inclined angle θ_1 , is measured by tak-

ing the reading of on the protractor, which is the intersection of the bottom of the protractor and the perpen-dicular bisectorθ_3. The fact that $\theta_1 = \theta_3$ is deduced by that: Due to parallel lines, $\theta_2 = \theta_3$, $\theta_1 + \alpha = \alpha + \theta_2$ and $\theta_1 = \theta_3$.
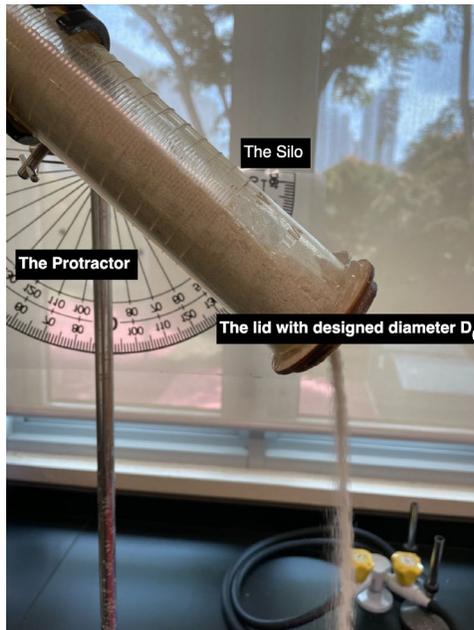

Figure 7. Sample experimental set-up with D_0=4mm and angle of inclination=45˚ (Author's Own)

Table 4. Raw data table for granular discharge rate through an inclined orifice

| The Angleθ (degree)	The Diameter of Orifice D_0 (4mm)			The Diameter of Orifice D_0 (8mm)			The Diameter of OrificeD_0(10mm)		
	Mass Before(g)	Mass After(g)	time(s)	Mass Before(g)	Mass After(g)	time(s)	Mass Before(g)	Mass After(g)	time(s)
0	125.68	116.59	9.08	414.55	202.18	20.48	193.57	113.24	4.35
10	87.17	79.49	8.67	221.13	136.75	8.34	167.23	83.43	4.79
20	79.49	67.86	13.46	323.14	221.13	11.02	115.72	76.3	2.41
30	63.72	52.60	13.76	423.96	343.12	9.24	157.02	53.15	6.56
40	104.81	85.86	25.12	210.94	137.24	9.11	204.08	79.73	8.36
50	85.86	83.34	4.05	422.95	342.97	11.50	165.21	127.49	2.91
60	79.99	76.63	6.43	342.96	294.77	7.84	190.08	114.28	6.38
70	74.54	70.20	10.75	294.77	269.95	4.88	114.28	54.86	6.47
80	76.63	74.54	7.13	485.52	468.84	5.16	176.34	118.97	8.29
90	/	/	/	465.30	456.70	3.06	183.21	76.81	17.26

18

Note that the region shaded grey is when no consistent flow is observed therefore data taken are invalid. All data is left in 2 decimal places, indicating the accuracy of by measurement depending on the equipment. The uncertainty of the balance is ±0.01, and the uncertainty of the stopwatch is ±0.01

Table 5. Sample Calculation and Uncertainty propagation

Sample Calculation	Uncertainty Propagation
Degree to radian: $10° \times \frac{\pi}{180} = 0.175(3s.f.)$ $\cos.175 = 0.985(3s.f.)$ Discharge Rate: $W = (87.17 - 79.49) \div 8.67 = 0.89gs^{-1}(2s.f.)$ Combined Variation in constants: $\frac{W}{W_0} = 0.89 \div 1 = 0.89$	Percentage Uncertainty: % uncertainty in mass difference +% uncer- tainty in time taken $(0.01 \times 2) \div (116.59 - 125.68) \times 100\% + 1.00 \div 9.08 \times 100\%$ $= 11.23\%(2d.p.)$ Absolute Uncertainty $11.23\% \times 1 = 0.11gs^{-1}$

Table 6. Processed data table for granular discharge rate through an inclined orifice (Author's Own)

	θ(degree)	θ (radian)	$\cos\theta$	$W(gs^{-1})$	$\frac{W}{W_0}$	Percentage Uncertainty (%)	Uncertainty
W_0	0	0.000	1.000	1.00	1.00	11.23	0.11
	10	0.175	0.985	0.89	0.88	11.79	0.10
	20	0.349	0.940	0.86	0.86	7.60	0.07
	30	0.524	0.866	0.81	0.81	7.45	0.06
	40	0.698	0.766	0.75	0.75	4.09	0.03
	50	0.873	0.643	0.62	0.62	25.49	0.16
	60	1.047	0.500	0.52	0.52	16.15	0.08
	70	1.222	0.342	0.40	0.40	9.76	0.04
4mm	80	1.396	0.174	0.29	0.29	14.98	0.04
W_0	0	0.000	1.000	10.37	1.00	4.89	0.05
	10	0.175	0.985	10.12	0.98	12.01	0.12
	20	0.349	0.940	9.26	0.89	9.09	0.08
	30	0.524	0.866	8.75	0.84	10.85	0.09
	40	0.698	0.766	8.09	0.78	11.00	0.09
	50	0.873	0.643	6.95	0.67	8.72	0.06
	60	1.047	0.500	6.15	0.59	12.80	0.08
	70	1.222	0.342	5.09	0.49	20.57	0.10
	80	1.396	0.174	3.23	0.31	19.50	0.06
8mm	90	1.571	0.000	2.81	0.27	32.91	0.09
W_0	0	0.000	1.000	18.47	1.00	23.01	0.23
	10	0.175	0.985	17.49	0.95	20.90	0.20
	20	0.349	0.940	16.36	0.89	41.54	0.37
	30	0.524	0.866	15.83	0.86	15.26	0.13
	40	0.698	0.766	14.87	0.81	11.98	0.10
	50	0.873	0.643	12.96	0.70	34.42	0.24
	60	1.047	0.500	11.88	0.64	15.70	0.10
	70	1.222	0.342	9.18	0.50	15.49	0.08
	80	1.396	0.174	6.92	0.37	12.10	0.05
10mm	90	1.571	0.000	6.16	0.33	5.81	0.02
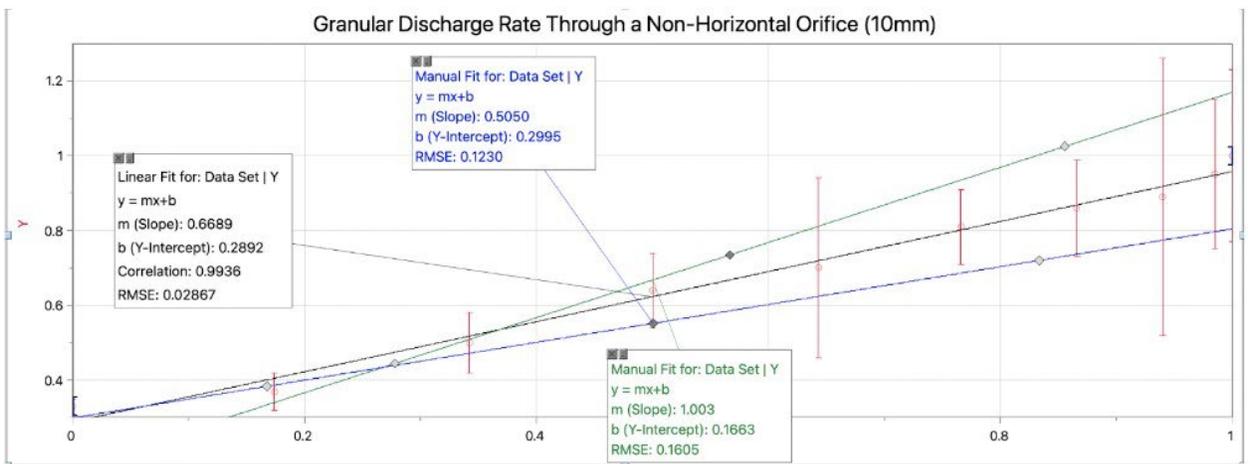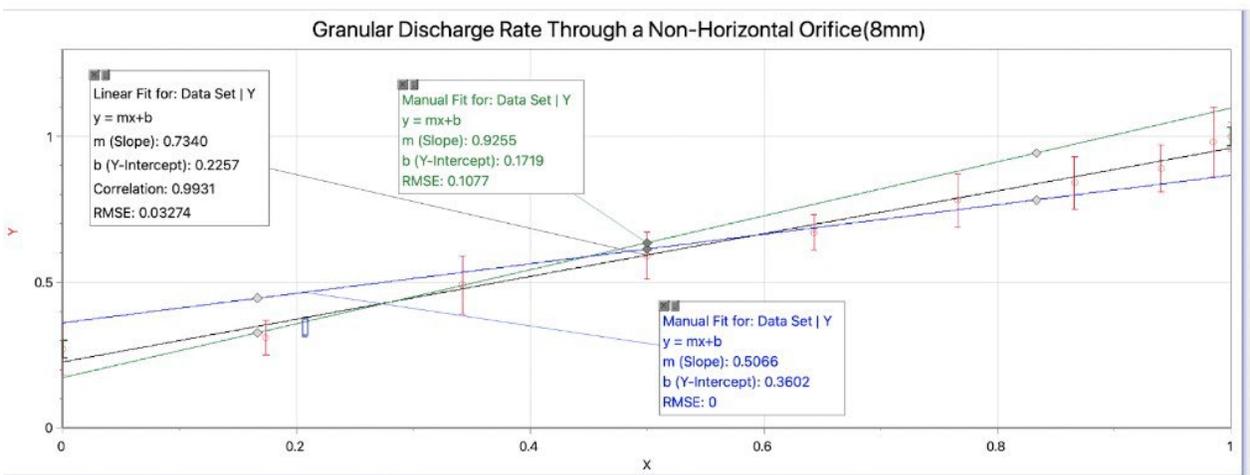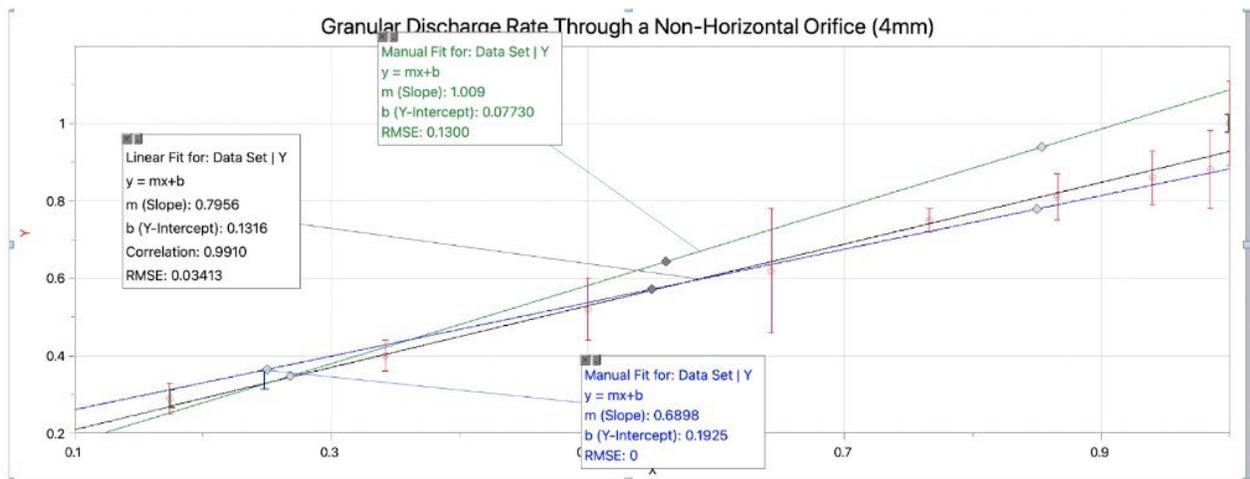
Graph 2. Experimental Data Visualisation, demonstrating the linearised and an- notated graph of the data points collected from granular discharge rate through a non-horizontal orifice experiment (from top to bottom the diameter of the orifice being 4mm, 8mm, 10mm; the black line represents the line of best fit; the green line represents the maximum gradient; and the blue line represents the minimum gradient (Author's Own)

Sample Data Processing from The Graph

For $D_0 = 4mm$

The equation of the line of best fit:

$y = 0.796x + 0.132 \ (RMSE = 0.0341; \ R = 0.991)$

The maximum gradient: 1.01 (3s.f.)

The minimum gradient: 0.690 (3s.f.)

Percentage Uncertainty $= (1.01 - 0.69) \div 0.796 \times 100\% = 40.2\%$

$= (1.01 - 0.69) \div 0.796 \times 100\% = 40.2\%$

For $D_0 = 8mm$

The equation of the line of best fit:

$y = 0.734x + 0.226 \ (RMSE = 0.0327; \ R = 0.993)$

The maximum gradient: 0.926 (3s.f.)

The minimum gradient: 0.507 (3s.f.)

Percentage Uncertainty $= (0.926 - 0.507) \div 0.734 \times 100\% = 57.1\%$

$= (0.926 - 0.507) \div 0.734 \times 100\% = 57.1\%$

For $D_0 = 10mm$

The equation of the line of best fit:

$y = 0.669x + 0.289 \ (RMSE = 0.0287; \ R = 0.994)$

The maximum gradient: 1.01 (3s.f.)

The minimum gradient: 0.505 (3s.f.)

Percentage Uncertainty $= (1.01 - 0.505) \div 0.669 \times 100\% = 75.5\%$

$= (1.01 - 0.505) \div 0.669 \times 100\% = 75.5\%$

Discussion of the Results

As shown from the visualisation and the numerical results, a strong linear relationship between $\cos\theta$ and $\frac{W}{W_0}$ can be interpreted, evidenced by the value of RMSE and R. This successfully substantiated my hypothesis. However, the uncertainty is relatively significant in all three experiments, and from the uncertainty propagation, I recognised that the main source of random error is still the time taken, and the modification in design after the preliminary investigation didn't successfully improve the measuring accuracy. This poses the need of more advanced digital measuring equipment, which is however hard to be achieved with the restriction of school lab conditions. Comparatively larger uncertainty is especially observed in the experiment with the orifice diameter of 10mm. Observing the raw data table, this can be rationalised by the significant decrease in the time frame chosen for the grain to discharge. As the packing height does not affect the discharge rate, the time frame is decided randomly in the experiment, suggesting a potential improvement in the following investigation, that a longer time frame, i.e. at least 20-second time frame should be selected to improve the accuracy of the measurement of time. In evaluation, the linear relationship is very strong with low systematic error. Despite the presence of random error, my hypothesis can still be supported and the value l, n in hypothesis obtained from the experiment can be used in prediction of the investigation of granular flow in a dynamic system.

Experimental Investigation of Granular Flow in Dynamic System

In this section, I will adapt the result obtained from section 3.2 to experimentally investigate the validity of the hypothesis (3.1.9) and (3.1.10), as written below: For $\omega t \in [2n\pi, \frac{\pi}{2}+2n\pi], n \in Z^{+}$

$$W = \frac{dm}{dt} = (g - \sin(\omega t)\omega^2 r)^{\frac{1}{2}}(\frac{W_0}{g^{\frac{1}{2}}})n + l(g - \sin(\omega t)\omega^2 r)^{\frac{1}{2}}(\frac{W_0}{g^{\frac{1}{2}}})\cos(\omega t) \qquad (3.3.1)$$

For $\omega t \in [2n\pi + \frac{\pi}{2}, 2\pi(n+1)], n \in Z^{+}$

$$W = \frac{dm}{dt} = (g - \cos(\omega t)\omega^2 r)^{\frac{1}{2}}(\frac{W_0}{g^{\frac{1}{2}}})n + l(g - \cos(\omega t)\omega^2 r)^{\frac{1}{2}}(\frac{W_0}{g^{\frac{1}{2}}})\cos(\omega t) \qquad (3.3.2)$$

The hypothesis will be verified with the silo with 4mm orifice, and the silo will be carried by a motor shown in Figure 8. From the linearised result in section 3.2, I have:

$$k = 0.796, \ b = 0.132 \ , W_0 = 1.00 (3\text{s.f.})$$

Figure 8. the motor used in the experiment(dismantled from a fan motor)

However, because the instantaneous discharge rate can't be measured directly, I decided to still measure the average discharge rate: the mass discharged in a certain time frame, defined from t_1 to t_2, To determine whether the equation can be integrated mathematically, the equation is drawn by Desmos of W against t. As the value of angular velocity and the length of the silo need to be known, using video analysis, I measured the angular velocity of the motor I will use in the experiment and the length of the silo. The angular velocity is measured through video analysis. The data calculated is presented below:

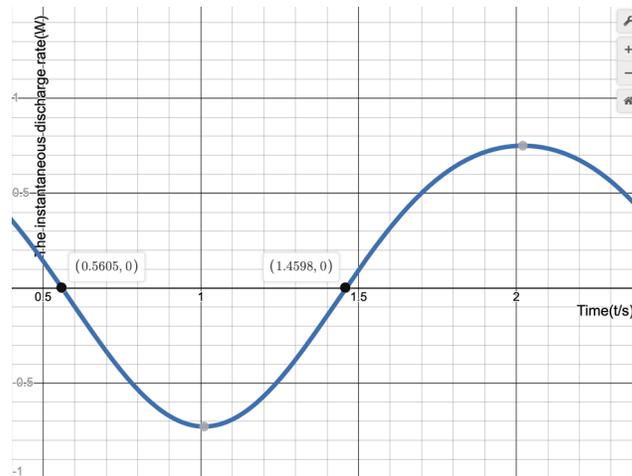$$\omega = 3.11 \, rads^{-1} \quad T = \frac{2\pi}{\omega} = 2.02 \quad r = 0.31m(2d.p.)$$

From the hypothesis (3.3.1), the domain for the first half of the equation is restricted to be:

$$[0, \frac{\pi}{2} \div \omega] = [0, 0.505](3s.f.)$$
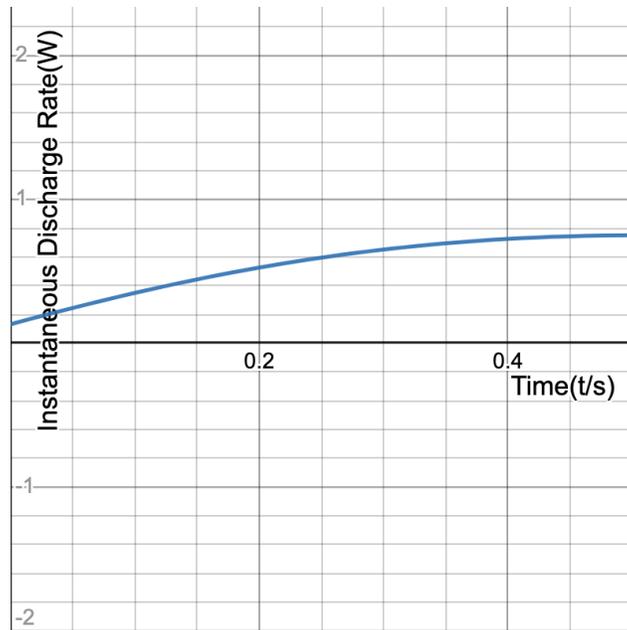
Graph 3. Visualisation of hypothesis (3.3.1), illustrated by Desmos)

The graph shows the function is continuous in the domain restricted. Hence, the discharge rate can be expressed as

$$m = \int_{t_1}^{t_2} [(g - \sin(\omega t)\omega^2 r)^{\frac{1}{2}}(\frac{W_0}{a^{\frac{1}{2}}})b + k(g - \sin(\omega t)\omega^2 r)^{\frac{1}{2}}(\frac{W_0}{a^{\frac{1}{2}}})\cos(\omega t)]dt \qquad (3.3.3)$$

Substitute the domain into $[t_1, t_2]$ and obtain

$$m = 0.275g \text{ (3s.f.)}$$

From hypothesis (3.3.2), the domain for the second half of the equation is restricted to be:

$$[\tfrac{\pi}{2} \div \omega, 2\pi \div \omega] = [0.505, 2.02] \text{ (3s.f.)}$$

Graph 4. Visualisation of hypothesis (3.3.2), illustrated by Desmos)

According to the graph,in the domains[0.505, 0.561] and [1.46, 2.02],the granular flow rate presented is positive. However, in [0.561, 1.46] the flow rate is negative, indicating that the grain is accelerating back in the silo, therefore the discharge rate is . As the figure proves that the discharge rate is a continuous function, the mass discharged for this domain is also expressed as equation (3.3.4)

$$m = \int_{t_1}^{t_2} [(g - \cos(\omega t)\omega^2 r)^{\frac{1}{2}}(\frac{W_0}{a^{\frac{1}{2}}})n + l(g - \cos(\omega t)\omega^2 r)^{\frac{1}{2}}(\frac{W_0}{a^{\frac{1}{2}}})\cos(\omega t)]dt \qquad (3.3.4)$$

Substitute the domain recognised into the equation's $[t_1, t_2]$
$$m = 0.319g \text{ (3s.f.)}$$

Therefore, the average granular discharge rate according to this hypothesis is predicted to be

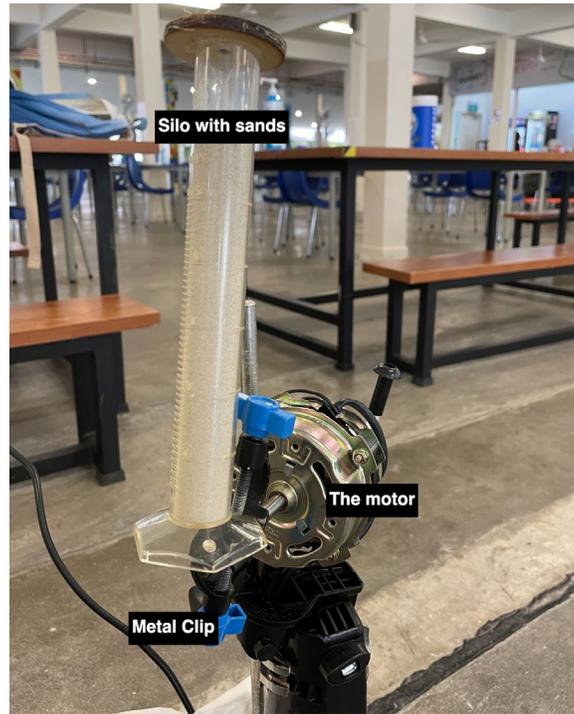$$(0.319 + 0.275) \div 2.02 = 0.294 gs^{-1} (3\text{s.f.})$$

Figure 9. Experiment set-up for the experimental investigation of granular flow in uniform circular motion

In this experiment, the silo is attached to the motor with a metal clip. The mass of the system is measured before and after the circular motion to calculate the mass discharged, and the time for the sands to discharge is taken at the same time. The experiment will be repeated for five times and the average discharge rate obtained from the experiment will be compared with the hypothesised value to quantitatively determine the validity of the equation in hypothesis. The size of the orifice and the angular velocity of the circular motion will be controlled throughout the repeats.

Table 7. Raw data taken for the experimental verification of the theory

The Initial Mass of the System (g)	The Mass of the System After Some Grain Discharged(g)	Overall Time Taken for the grain to be discharged(s)
97.45	79.53	47.36
76.31	59.46	42.31
89.21	73.00	43.29
96.34	84.06	32.47
75.06	60.45	39.79

Table 8. Sample Calculation and Uncertainty propagation

Sample Calculation	Uncertainty Propagation
The discharge rate: $(97.45 - 79.53) \div 47.36 = 0.38$ (2d.p.)	Percentage Uncertainty: $(0.38 - 0.29) \div 0.29 \times 100\% = 28.7\%$

Table 9. Percentage nncertainty calculation

Discharge Rate gs^{-1}	percentage error %
0.38	28.7
0.40	35.5
0.37	27.4
0.38	28.6
0.37	24.9
Average Discharge Rate gs^{-1}	percentage error %
0.38	29.0

The value obtained from theoretical calculation deviates by 29% from the theoretical value. This could potentially indicate the accuracy of the theoretical hypothesis is highly limited. However, observing the data, the percentage errors are in a range between 24.9% and 35.5%, suggesting a significant systematic error is presented in the equation. Moreover, the theoretical graph of granular discharge rate presents a large domain where the discharge rate is 0, whereas from my observation, the grain is discharged continuously throughout the circular motion. As the experiment only considered the possible acceleration in vertical vibration, the possible presence of horizontal vibration should be analysed further, which may transfer some momentum to the grain (Evesque and Meftah, 1805). Moreover, sand liquefaction is potentially generated by the circular motion, resulting in a Brownian motion inside the silo, whereas the equation is derived from static analysis, posing another potential limitation in the theory.

Discussion

As the importance of investigating the characteristics of granular materials in helping scientists and engineers to understand the fundamental mechanism of most industrial materials is arising, it's essential to understand the change in granular flow rate in a dynamic system. Centring on the condition of uniform circular motion, this investigation developed its unique mathematical equation to predict the granular flow rate, based on Beverloo's equation, the equation effec- tively predicts the granular flow rate in a static condition, and two hypotheses summarised from prior literature. By experimentally testing the mathematical equation, this study found that the first hypothesis of the change in effective diameter as a result of the formation and break up of static arch is valid, and its impact is reflected in the linear relationship between $\cos\theta$ and the product of the constants C and k. However, a large percentage deviation is discovered when the hypothesis is applied to the dynamic system. This suggests that the second hypoth- esis of manipulating classical force analysis into granular material has a significant weakness, especially when it's merged into the overall application. Furthermore, the large percentage uncertainty com- pared to mathematical prediction potentially reflects that the first hypothesis of effective diameter could also have some weakness when it's applied to a more complicated condition, such as a constant and rapid change in inclined angle, which could potentially disrupt the inner distribution of granular material and hence weakening the founda- tion of Beverloo's equation. These problems proposed above suggest significant reliability of experimental investi- gation when researching granular material under more complicated conditions, and considerable weakness of mathe- matical prediction in analysing some unpredictable phenomena. Hence, the investigation partially resolves my ques- tion about the relationship between a constant change in angle and granular flow rate, while some questions still remain unanswered. Nonetheless, the mathematical equation I proposed in the investigation is still recognised as a good precedent for further research and refinement, considering the percentage error is relatively consistent. Some potential implications in the experiment and theory might have caused this.

Implications

In this investigation, there are some considerable implications in both the mathematical theory and experiment. Firstly, this investigation found that the research on the flow rate of granular material requires highly precise meas- urement and variable control, as the condition of granular material can be easily affected by the outer environ- ment. Although I attempted to maintain consistent laboratory conditions and use precise apparatus, the equipment and conditions pro- vided by the school could hardly meet the professional laboratory requirements, resulting in a consistent random error in my investigation. Hence, there's also a possibility that my mathe- matical hypothesis is solid, but my lab condition restricts the value I experimentally to have a consistent correspondence with the actual theory. Secondly, the assumptions I discussed before formulating my hypothesis could in reality have some fundamental impact on the validity of my mathematical hypothesis, which is to be determined by the experiment. Although there is a possibility that Beverloo's relation is fundamentally flawed in a more complicated dynamic sys- tem, the consistent percentage error tested in the last experiment suggests that this possibility is relatively low. Hence, further research possibilities will be discussed in the next section.

Further Research

Based on the implications of my investigation, I will propose the following further research opportunities con- centrating on the investigation of granular flow in a dynamic system. Firstly, although this experiment rationalises the role of the constant in Beverloo's equation, except for the change in angle, it's still unclear in what other conditions the constant might change. Hence, further investigation dedicated to explaining the constants in Beverloo's equation is highly encouraged, as it solidifies the foundation of our understanding of Beverloo's equa- tion. Moreover, in dealing with the granular flow in a dynamic system, observation of the inner distribution of the granular material is essential. Observation and investigation of potential sand liquefaction and its impact

on the static arch and effective diameter are necessary prior to more complicated research. After a considerable amount of experimental observation, computer simulations could be a useful tool for studying granular material, as they could effectively visualise the inner dynamics of granular material.

Acknowledgments

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