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An In-depth Analysis of Symmetries and their Implications in Physics

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ABSTRACT

The concept of symmetry is essential in understanding classical and quantum physics. Symmetry describes a system that remains unchanged in structure and behavior after undergoing a transformation. In this paper, I will describe the implications of symmetries (global and gauge) in Newton's laws of mechanics, Maxwell's electromagnetism equations, and quantum particle physics – in particular the Higgs mechanism – with the help of Noether's Theorem. Purely based on symmetrical elements, this paper will then determine the isospin composition of pions given certain restraints. This solution will connect findings in preceding theories to current and future studies relevant to the subject of symmetry in physics.

Introduction

Symmetry is often conceptualized as a rudimentary element, usually found in simple geometry and patterns. Although these are adequate examples and applications of symmetry, there is a far greater role for symmetry in the physical sciences that helps describe the behavior of the physical world. Symmetries are present in biology, chemistry, astronomy, even the humanities and arts, however, they are fundamentally rooted in physics.

Fundamentally, a system is said to possess a symmetry if it remains invariant after a transformation. This idea has infiltrated many essential topics throughout classical theoretical physics from as early as Newton's laws of mechanics. The conservation laws of energy and momentum, initially emerging from Newton's Laws, were later credited to symmetries by Emmy Noether. In 1905, Noether produced and proved the Noether Theorem: For every continuous symmetry of the laws of physics, there must exist a conservation law and associated conserved quantity. For every conservation law, there must exist a continuous symmetry. We will review this theorem, along with the Principle of Least Action in more detail.

So far, the symmetries mentioned have been independent of the observer's frame of reference or location in space-time and only involve classical laws of physics. Simply put, these types of symmetries are called global and classical symmetries. Diverging from global, classical symmetries, gauge theory allows for local symmetries that can be applied differently in different points in space. To highlight the similarities and nuances of gauge symmetries, we analyze them in more detail in the context of more familiar theories such as Newtonian gravity and electromagnetism.

The significance of gauge symmetries was further highlighted, by the discovery of the Higgs boson in 2012, a fundamental particle associated with the Higgs field. Upon further observation, the Higgs field arises from the breaking of a gauge symmetry involving the interaction of the Higgs particle with the other fields in the standard model of particles. This idea, along with symmetry breaking, is explored in a later section.

Symmetry is also present at the most fundamental level of physics with quarks and particle physics. Initially, it was assumed that there were only three species of quarks, named up, down, and, strange, enough to account for the properties of an entire class of particles using various combinations of different quarks. In recent decades, physicists have discovered up to six different species, labelled c, b, s, t, which brings additional dimension to symmetry groups. This explains existing particles, such as hadrons, and predicts new particles. Unfortunately, these new species of quarks do not solve the problem of quark confinement. These new species of quarks form a complete SU(3) symmetry group that relates to the quark confinement problem – single isolated quarks are not found in nature – thorough mandating the formation of color singlets. The quark confinement problem can be reformulated in terms of this newly completed SU(3) symmetry group. Confinement mandates the groping of quarks in color singlets, where 2 color is the charge of each quark under this new gauge symmetry.

At the end of this paper, I will use the governing principles of symmetry to deduct information about hypothetical particles postulated by a new hypothetical phenomenon in physics. These powerful rules can verify or reject different possible models fundamental particles.

Governing Principles of Symmetry in Physics

Principle of Least Action

To understand Noether's Theorem, we must first discuss the Principle of Least Action, an extremum principle. Let's say we have a free-moving particle in a gravitation field that that moves from point A to point B in a certain amount of time. The Lagrangian (the difference between the particle's potential and kinetic energy), a function of time, can be integrated over the time duration of the path to obtain the action, $S = \int_{t_1}^{t_2} L\,dt$. This equation properly explains the action of the particle moving by free motion, only acted on by the inertial force.

This integral gives different values for each path that the particle takes; however, there exist a path which minimized the value of S. The principle of least action states that a system (or particle) always evolves (takes the path) which minimizes this integral. This principle is the reason why light always travels in a straight line through a medium of constant refractive index. Now if we introduce a second medium with a different refractive index on the lights original path, it breaks and takes a new non-straight trajectory. This new trajectory now minimizes the integral for our new two medium problem.

Statement of Noether's Theorem

Suppose we have a system that satisfies the equations of motion (the first derivative of its path x is equal to its velocity v and the second derivative is equal to its acceleration a). In classical mechanics, there are three translations that can be done on a system that will keep satisfying the equations of motion: translations in space, rotation, and time.

The Lagrangian, a function that describes the state of a dynamic system, can be defined in terms of position coordinates and their time derivatives, $L = L(q_i(t), \dot{q}_i(t))$.

Now, let's change our current paths $(q_i(t))$ into different paths $(q_i'(t))$. This translation is a symmetry of the Lagrangian if $L\big(q_i(t), \dot{q}_i(t)\big) = L(q_i'(t), \dot{q}'_i(t))$ because they correspond with symmetries of the equations of motion.

Noether's Theorem states that "For every continuous symmetry of our Lagrangian, we have a conserved quantity along trajectories that satisfy the equations of motion." Translations in space correspond with the conservation of total momentum, translations in rotation correspond with the conservation of angular momentum, and translations in time correspond with the conservation of energy.

Gauge Symmetries

In addition to global symmetries, we can analyze local symmetries with gauge theories. To gain an intuitive sense of what a local gauge symmetry is, let us use an economic model. If all monetary values were re-scaled by the same factor, our economy would not change (ignoring psychological implications), giving an example for a global symmetry. However, each country can scale their economy independently of other countries. We establish exchange rates (a gauge field), making this scaling symmetry a local symmetry. Because of fluctuating exchange rates, investors will follow the route that make them the most money, as if being acted on by a force, which will further influence the fluctuation. Physically speaking, gauge fields both affect and are affected by systems in local symmetry.

In terms of applications, we can view gauge symmetry in terms of potentials. We know that the formula for the gravitational potential energy near the surface of the Earth is $V(h) = mgh$. Yet, this specific value, relies on an arbitrary initial reference frame, offers no physical significance. Instead, we focus on the change in gravitational potential energy: $\Delta V = V(h_1) - V(h_2)$. In other words, gauge symmetries are redundancies of the system and describe multiple copies of the exact same state. This is like the case of monetary exchange rates. If all countries experienced the same amount of economical inflation, then the path taken by businessmen would not change. That is equivalent to changing the off set in our gravitational potential. What matters are the local variations in potential (or monetary exchange rates).

We can also look at the electric potential using a capacitor (two plates that store electric energy in an electric field). The equation for electric potential, $-\Delta V = qE\Delta x$, can be manipulated into $-\frac{dV(x)}{dx} = qE$. To test this application of gauge symmetry, let us now add a constant V_o for the following equation:

$$-\frac{d}{dx}(V(x) + V_o) = -\frac{dV(x)}{dx} - \frac{dV_o}{x} = qE$$

As expected, the equation remains invariant to changes in the system. The constant V_o only helps describe an infinite number of unphysical states with the same equation of motion and different potential energies. However, there is only one physical degree of freedom that matters.

Implications of Symmetry in Particle Physics

Symmetry Breaking and Mass Creation

In this section we are going to talk about how breaking a symmetry in quantum mechanics can lead to consequential phenomena in physics. We talk about how the breaking of the weak nuclear symmetry results in all the particles in the universe acquiring a mass.

When looking at the role of gauge fields on the weak nuclear force, it is a bit more complicated. These systems are based on gauge symmetries that transforms electrons and neutrinos into one another, but this changes the systems and breaks the symmetries. To maintain the theory of gauge symmetry for the weak nuclear force, we turn to the Higgs Mechanism and spontaneous symmetry breaking.

Essentially, we are manipulating the system so that it appears to be broken without being broken. A common example is a ferromagnetic material. At high temperatures, all the magnets point in random, different directions, making all directions look the same. At low temperatures, the magnets point in the same direction, seemingly breaking directional symmetry. Only the lowest energy state breaks the symmetry. The symmetry is still present in the fundamental equations of the system.

Let's do the same to the gauge symmetry of the weak nuclear force with a scalar Higgs field. We can give it potential energy so that the lowest energy state is asymmetrical. The potential energy is rotationally symmetric, but the rotational symmetry does not rotate real space, but rather the field space in which the Higgs field lives. A state of lowest energy (the vacuum state) occurs at the bottom of the potential. If we rotate the system (including the vacuum state) around the origin, the theory is invariant, and the symmetry is still there, even if it does not appear so from the vacuum state. As a result, when the Higgs field spontaneously breaks the gauge symmetry of the weak nuclear force, it gives mass to the W+, W and Z bosons.

To better understand the concept of spontaneous symmetry breaking, let us follow the derivation very closely to solve a problem involving a particle (mass m) on a vertically rotating ring of radius R and constant angular velocity ω as shown in Figure 1.

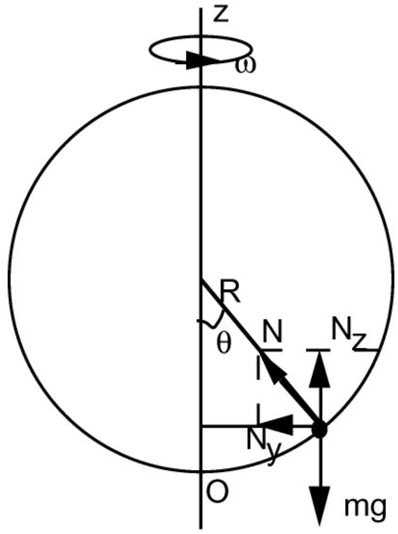

Figure 1. Particle on a Ring with Labelled Values

If the particle remains at angle θ, we can write the following equations for the components of force N.

$$N_x = Ncos\theta = mg$$

$$N_y - ma = (N\omega^2 R)sin\theta = 0$$

Next, we gain an expression for the equilibrium angle θ_o.

$$Ncos\theta = m\omega^2 Rcos\theta = mg$$

$$cos\theta_o = \frac{g}{\omega^2 R}$$

Through further analysis of the above equations, we can conclude that there is a critical ω where the solution does not apply: $\omega = \sqrt{\frac{g}{R}}$.

Let's now take a Lagrangian approach with the variable θ. Because $v_\theta = R\dot{\theta}$ and $v_\phi = \omega Rsin\theta$, we can write the Lagrangian as $L = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m\omega^2 R^2 sin^2\theta - mgR(1 - cos\theta)$.

We can also express this Lagrangian in terms of an effect potential V_e that describes the conflicting relationship between the gravitational force and the repulsive centrifugal effect: $L = \frac{1}{2}mR^2\dot{\theta}^2 - V_e$. Then, we'll introduce two additional dimensionless equations to express V_e and measure the centrifugal effect, respectively: $U = \frac{V_e}{mgR}$ and $\beta = \frac{\omega^2 R}{g}$. The extrema of U will occur at $\frac{\partial U}{\partial \theta} = sin\theta(1 - \beta cos\theta) = 0$.

Finally, we can plot the potential U as a function of θ and β as shown below. This graph allows us to uncover insight regarding spontaneous symmetry breaking. At θ = 0, the most symmetric state with least potential energy, the state becomes unstable when the frequency of rotation (β) is dramatically increased. This is a great description of what occurs with particle theory involving massless bosons. The initial stable symmetric state becomes unstable, producing massive bosons as a result.

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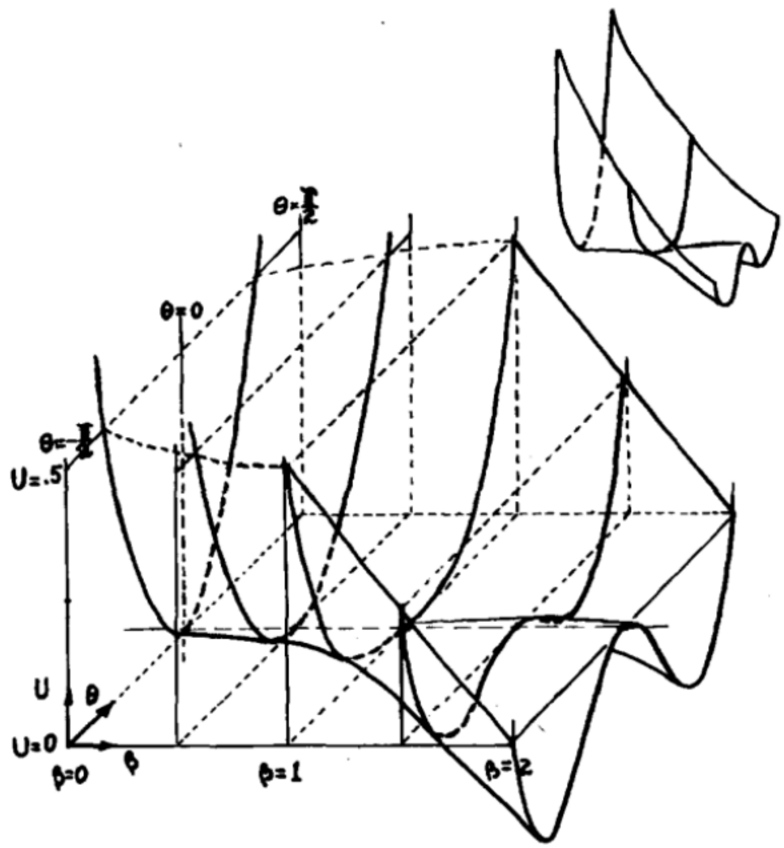

Figure 2. The potential U graphed as a function of θ and β

The Quark Model

Digging deeper into particle physics, we can now discuss symmetries and the quark model. It turns out that the mass and nuclear force of protons and neutrons are nearly symmetrical. In the 1930's, Heisenberg proposed the nucleon, a single particle in which the proton and neutron are two states of. This influenced a new idea: the isospin, a conserved quantity of strong interaction. (The proton has an isospin of positive 1/2 while the neutron has one of negative 1/2.) Since protons and neutrons originated from the same particle, interactions between two protons, two neutrons, or a proton and a neutron were nearly the same. Within the quark model, we can think of this symmetry as being a symmetry between the u and d quarks.

Moreover, this discovery led to the introduction and classification of baryons, now recognized as the building blocks of particle physics. To be exact, there are eighteen particles divided into the octet and decuplet (also known as the Eightfold and Tenfold Ways, respectively).

Each baryon consists of a combination of three lightweight quarks with possible flavors: up, down, and strange. Quarks are also spin-1/2 particles. As a result, a baryon must have a final spin of 1/2 or 3/2. For a baryon, it turns out that the combination of flavor and spin must be completely symmetric.

In terms of notation, let's represent a flavor state as Ψ and a spin state as S. Additionally, ψ_{ab} and S_{ab} represent the flavor and spin states that are antisymmetric for particles a and b. For the first eight baryons in the octet, a symmetric flavor-spin combination looks like $\psi = \psi_{12}S_{12} + \psi_{23}S_{23} + \psi_{13}S_{13}$.

For example, a proton with two up quarks and one down quark is a superposition of an antisymmetric quark flavor state and an antisymmetric spin-1/2 state to produce an overall symmetric state of $p = (udu - duu)S_{12} + (uud - udu)S_{23} + (uud - duu)S_{13}$.

The construction of the remaining seven baryons in this group are very similar in that each particle is also composed of a different symmetric state.

Like the octet, the decuplet also consists of baryons with completely symmetric flavor states that share similar properties: spin-3/2 particles with electric charge and strangeness (which describes particle decay).

Quark Confinement

With three initial species of quark, the properties of an entire class of particles are accounted for. In theory, quarks are mighty particles. Unfortunately, free quarks have yet to be found. If quarks do not show themselves in the open, scientists created theoretical assumptions about the forces that bind them together and are now charged with explaining their confinement within the particles they make up.

Quarks make up particles called hadrons, divided into two large subgroups of baryons and mesons. One recent model to explain quark confinement and attempts to calculate known properties of the hadrons has been proposed by Kenneth Johnson. Johnson suggests that the quarks are trapped inside a bag, without the ability to penetrate but can exert pressure from the inside. Because the bag's energy is proportional to its volume, potentially unlimited amounts of energy are required to separate the quarks. The system reaches equilibrium when the bag's tendency to shrink is balanced by the pressure of the quarks inside. With this model, we can compute various properties of hadrons with reasonable accuracy.

Question

In this section, we demonstrate with an example how the rules of symmetry can guide us in making prediction about unknown process and physics. Not only can these rules make predictions about the unexplored, but also, they can help us eliminate the redundant particles and quantum states that a theory might predict. For example, the selection rules that are based on molecular symmetry groups play a role in determining the products of chemical reactions.

To illustrate the workings of these selection rules, let's assume a pair of pions with zero orbital angular momentum were produced in a collider reaction. The selection rules can help us determine what total isospin values are possible for this system. Knowing the total isospin brings us closer to deciphering the flavor content of the quarks that are confined within the pions.

In the absence of orbital angular momentum for a system, the total angular momentum of the system is simply equal to its spin: $J_{total} = J_{orbital} + S_{total} = 0 + S_{total} = S_1 + S_2$. Now, since pions are spin zero particles, we conclude that $S_{total} = 0 + 0$ is a symmetric s-wave state – symmetric implies the wave function remains unchanged under the exchange of the two pions. Next, note that the total wave-function of the two pion system can be written as $|\psi_{1,2}\rangle = |Spin_{1,2}\rangle \times |Isospin_{1,2}\rangle$.

For our symmetric – the system cannot distinguish between the two pions – system of two pions, a symmetric spin wave-function constraints us to pick a totally symmetric isospin wave-function as well.

The resolution now lies in finding symmetric isospin states of a two-pion system. Pions are isospin - 1 particles and we have to use the appropriate rules concercing the addition of isospin when considering a two pion system. The total isospin of the system can take up values in $\{0, 1, 2\}$. The state with highest total $J_3 = m_{total}$ value is a symmetric combination of the tensor product of the two pion isospins.

$$|J_{total} = 2, m_{total} = 2\rangle = |J_1 = 1, m_1 = 1\rangle \otimes |J_2 = 1, m_2 = 1\rangle$$

Now it is worth noting that the lowering and raising operators do not change the symmetry of the states, so all the states with $J_{total} = 2$ are also symmetric. To determine the symmetry of the states with $J_{total} = 1$, we notice that there are only two possible combinations of pion isospin that lead to $m_{total} = 1$, the highest m value for states with $J_{total} = 1$:

$$|m_{total} = 1\rangle = |m_1 = 1\rangle \otimes |m_2 = 0\rangle \pm |m_1 = 0\rangle \otimes |m_2 = 1\rangle$$

The plus sign corresponds to the state $|J_{total} = 2, m_{total} = 1\rangle$ because it is symmetric. Thus, we conclude that states with $J_{total} = 1$ are antisymmetric.

The singlet state $|J_{total} = 0, m_{total} = 0\rangle$ is a combination of

$$|J_{total} = 0, m_{total} = 0\rangle = a \cdot |m_1 = 1\rangle \otimes |m_2 = -1\rangle + b \cdot |m_1 = -1\rangle \otimes |m_2 = 1\rangle + c \cdot |m_1 = 0\rangle \otimes |m_2 = 0\rangle$$

The first two terms are symmetric w.r.t. $1 \leftrightarrow -1$, so the singlet in question also has symmetric properties. Therefore, the following categorization holds:

$$\underbrace{\begin{array}{c} |J_{\text{total}} = 2, m_{\text{total}} = 2\rangle \\ |J_{\text{total}} = 2, m_{\text{total}} = 1\rangle \\ |J_{\text{total}} = 2, m_{\text{total}} = 0\rangle \\ |J_{\text{total}} = 2, m_{\text{total}} = -1\rangle \\ |J_{\text{total}} = 2, m_{\text{total}} = -2\rangle \end{array}}_{\text{Symmetric}} \quad \underbrace{\begin{array}{c} \\ |J_{\text{total}} = 1, m_{\text{total}} = 1\rangle \\ |J_{\text{total}} = 1, m_{\text{total}} = 0\rangle \\ |J_{\text{total}} = 1, m_{\text{total}} = -1\rangle \\ \\ \end{array}}_{\text{Anti-symmetric}} \quad \underbrace{\begin{array}{c} \\ \\ |J_{\text{total}} = 0, m_{\text{total}} = 0\rangle \\ \\ \\ \end{array}}_{\text{Symmetric}}$$

Figure 3. Categorization of symmetric and antisymmetric states

In summary, the concept of symmetry has further applications than what has been currently explored. The rules of symmetry have the potential to uncover predictions in theoretical particle physics.

Conclusion

In this paper, we have covered the applications and implications of symmetries throughout various topics across physics. We discussed global, classical symmetries relating to classical mechanics using the Principle of Least Action. With a Langrangian integral for the action, we stated the implication of Noether's Theorem (relating symmetries to conservation laws of motion). We explored local symmetries using gauge fields and real-world analogies, discovering its presence in electromagnetism and gravity. Although gauge fields seemed to reach a limit with the weak nuclear force, we used the Higgs field and spontaneous symmetry breaking to keep the theory intact. We looked at symmetries in particle physics, specifically with quarks. We highlighted the problematic nature of quark confinement, but also mentioned recent models proposed by current theoretical physicists. Lastly, we explored a hypothetical problem involving a pair of pions in the absence of orbital angular momentum. Applying the knowledge gained throughout the paper regarding symmetries, we produced the possible isospin values for this system. The world of theoretical physics is still on the search for new symmetries, with the hope to explain additional mysterious features of our universe.

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