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An Estimation of Hailstone Radii Using the Collatz Conjecture

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ABSTRACT

There is an alleged connection between a much researched, yet unsolved math problem, the Collatz conjecture, and a quite common, damaging meteorological phenomenon: hailstorm. The relationship would be that, in sequences generated by Collatz algorithm, the way in which numbers rise and fall resembles hailstones going up and down inside a cloud, whence the name "Hailstone sequences". The aim of this paper is two-fold: first, to use JavaScript to research on the Collatz conjecture with the perspective of a high-school student. Our algorithm tested a generalized form of the conjecture for multiple primes (p =3, 5, 7) and signs (+,−). The Pearson correlation coefficient found between the initial value and, respectively, the total stopping time or the maximum value reached excluded any linear correlation. The second (and main) goal was to assess the hypothesis whether hailstones could indeed follow a Collatz-like function trajectory, studying the implication on the radii of them. Introducing the concept of *conversion formula,* we estimated the final radii for different functions (straight line, square-root, square, logarithmic, exponential), unit of measures (from Km to mm), and starting heights ranging from 4000m to 10000m, should the motion of hailstones behave like a Collatz function. In all but one case, we did not get radii believable in size, and *reasonably* randomly distributed. For the linear formula (in cm), the χ^2-test values between our estimated values and Nelson's model values are above the critical values. Hence, we should reject the initial hypothesis.

Introduction

The Collatz conjecture is one of the most famous, researched, unsolved problems in mathematics and enjoys the appealing property, often attributed to celebrated number-theoretic questions, of being simple to state and apparently impossible to answer. The problem can be formulated as follows: consider the integer-valued function defined as

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ 3n+1 & \text{if } n \text{ is odd,} \end{cases} \qquad (1)$$

where n is any positive integer number. The Collatz conjecture states that, no matter what the starting positive integer n is, the function iterates of $f(n)$ eventually reach the value 1.

Stated in 1937 by Lothar Collatz, the conjecture has been checked for all values up to 2^{68} and yet no proof has been found as of 2022. "Mathematics may not be ready for such problems" Paul Erdos stated. Despite this pronouncement, the Collatz conjecture keeps attracting the interests of many mathematicians, and the study of the problem has not been without reward, one of the most recent of which by (Tao, 2019).

There are already survey papers on the topic, intended either for high-level mathematicians (Lagarias, 1996) or for undergraduate students (Lohia, 2022). We aim to complement the abstractness of those papers following the perspective of a high-school student who approaches this question for the first time. This means discussing naïve approaches and why they do not work, concrete examples, and general Collatz-like functions for multiple primes (p =3, 5, 7) and signs (+,−), using JavaScript algorithms and computations. In particular, as a clue of the hardness of the conjecture, we compute the Pearson correlation coefficient between the initial value n_0 and, respectively, the total stopping time and the maximum value reached, concluding that there is no linear correlation among them.

Besides this, we aim to spark interest in real world applications of the Collatz conjecture to adopt different perspectives of this unsolved problem and gain insights from them. Indeed, as a second and main goal of this paper, we address the alleged connection between the sequences of numbers involved in the iteratives of the function $f(n)$ and the trajectory of a hailstone inside a cloud. Namely, the values seem to rise and fall multiple times, somewhat analogously to hailstones during their formation: this has led people to call Collatz sequence also "Hailstone Sequences" or "Hailstone Numbers" (Pickover, 2001). We study to which extent this is an accurate description and what the implications would be if the formation of hailstones indeed followed a Collatz-like function pattern. The hypothesis we want to test is that there exists a *conversion formula* (see *The analogy with the Conjecture* Section) such that the height with regards to the ground of hailstones can be described using that formula and the iterations of Collatz algorithm. We focus on the consequence of that on the size of radii of the final hailstones. Inspired by real-life measurements, we introduce three *viable criteria* that results from our simulations should meet (see *Study Method* Section). We test linear, square-root, square, logarithmic, and exponential conversion functions, with height expressed in different unit of measures (from Km to mm) and ranging from 4000m to 10000m. In all but one case, the linear formula (in cm), we need to reject our initial hypothesis. Finally, for the only viable case, we do the χ^2-tests between our estimated values and Nelson's model data (Nelson, 1983), and find that the χ^2-values are always above the respectively critical values. Hence, also in this case we should dismiss the initial hypothesis, even if further steps into the study should be taken into consideration.

Why is the Collatz conjecture so hard?

A lot of different approaches were used to understand better the sequences that Collatz algorithm produces. Dealing with a statement involving positive integers, a natural approach would be to use strong induction: indeed, for the starting number 1 the problem is automatically true, so we need to check the inductive step. If we assume that the Collatz algorithm reaches the value 1 for all the integers less or equal than n, if n is odd then $n+1$ is even and $(n+1)\,/\,2$ is less or equal than n: so by induction the Collatz sequence starting at $n+1$ converges as well.

Now assume that n is even: then $n+1$ is odd and so $f(n) = 3n+4$ and $f^{(2)}(n) = (3n+4)/2$ which is never less than or equal to n. However, $(3n+4)\;/\;4 < n$ for any integer greater than 4. After checking for $n = 2,3,4$ that the conjecture still holds, we can conclude that if $f(n)$ is divisible at least by 4, Collatz converges also starting at $n+1$. If it is not the case, then $f^{(2)}(n)$ would be odd and

$$f^{(3)}(n) = 3\left(3\frac{n}{2}+2\right)+1 = 9\frac{n}{2}+7. \qquad (2)$$

More precisely, using the following equivalent way to write the Collatz function $f(n)$

$$T(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{3n+1}{2} & \text{if } n \text{ is odd,} \end{cases} \qquad (3)$$

by induction one can prove (Lagarias, 1996, p. 8) that the $k-th$ iteration $T^{(k)}(n)$ can be expressed as

$$T^{(k)}(n) = \frac{3^{\sum_{i=0}^{k-1} x_i(n)}}{2^k} n + \sum_{i=0}^{k-1} x_i(n) \frac{3^{\sum_{j=i+1}^{k-1} x_j(n)}}{2^{k-i}} \qquad (4)$$

where $x_i(n)$ reflects the parity of the $i-th$ iteration $T^{(i)}(n)$, assuming value 1 if $T^{(i)}(n)$ is odd, and value 0 if $T^{(i)}(n)$ is even. However, at first sight the equation $T^{(k)}(n) < n$ seems hard to solve. To understand at which extension this perception is true, we can try to analyze the behavior of $f^{(k)}(n)$ (and therefore of $T^{(k)}(n)$) via software[1]. We considered *Collatz-like function* of the form

$$f_{p,+}(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ pn+1 & \text{if } n \text{ is odd,} \end{cases} \quad \text{or} \quad f_{p,-}(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ pn-1 & \text{if } n \text{ is odd,} \end{cases} \qquad (5)$$

With p = 3, 5, or 7 respectively, and for n ranging from 1 to 10000 and for each choice $(p, n, sign)$, we calculated:

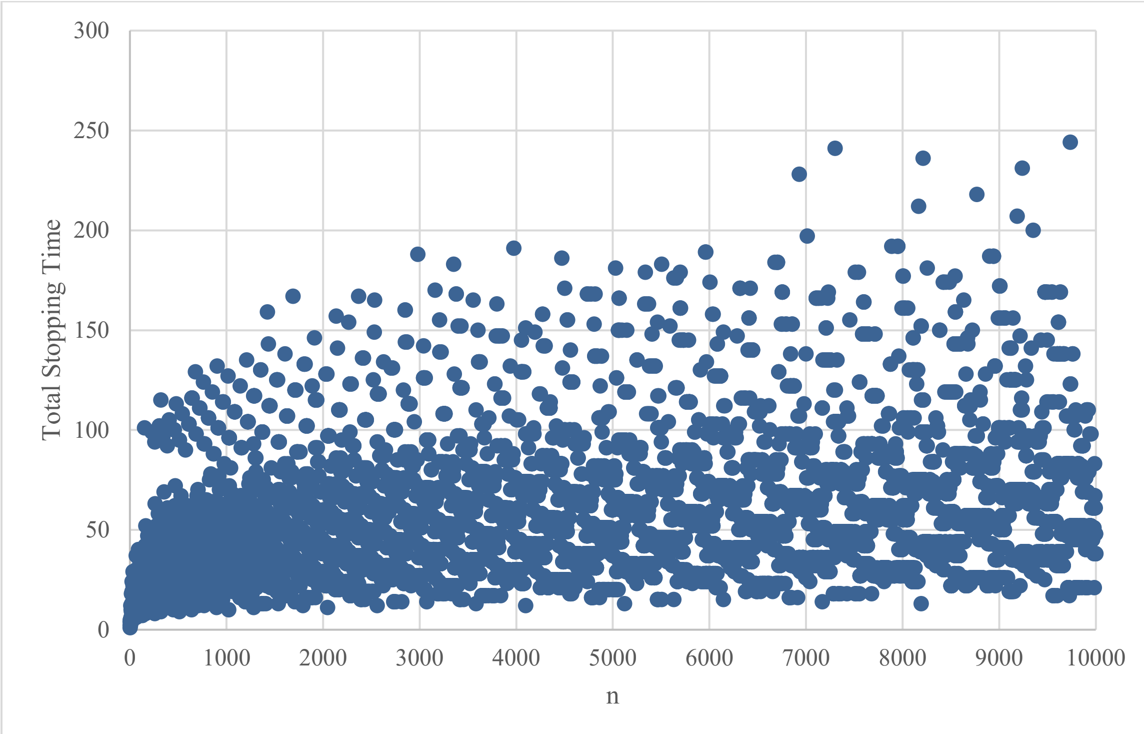
a) whether it generates a sequence which reaches 1 (type 1), it forms a cyclic sequence which does not pass by the value 1 (type 2) or if it exceeds the value 2147483647 (this value is a Java limitation) (type 3);
b) the number of iterations needed to reach 1 (type 1), called the *total stopping time* (Lagarias, 1996, p. 5) or even form a cyclic sequence which does not pass by value 1 (type 2) and denoted by $\sigma(p, n, \pm)$ for a fixed prime p and the sign, if n is of type 1 or 2;
c) fixed the prime p and the sign, if n is of type 1 or 2, the maximal value reached by the sequence $\{f^{(k)}{}_{p,\pm}(n)\}_{k\geq 0}, denoted\ by$ $\max(p, n, \pm)$;
d) fixed the prime p and the sign, the (sample) Pearson correlation coefficient [2] $r_{n,\sigma}$ between the initial number n and the total stopping time;
e) fixed the prime p and the sign, the (sample) Pearson correlation coefficient $r_{n,max}$ between the initial number n and $\max(p, n, \pm)$;
f) the mode, the mean, and the standard deviation of the total recurring stopping.

Table 1: Results from the generalised Collatz functions using Java algorithm[3]

$(p, n, \pm)$	3n+1	3n-1	5n+1	5n-1	7n+1	7n-1
How many $n's$ of type 1	10000	3244	256	1120	82	13 (only 2^n)
How many $n's$ of type 2	0	6756	470	0	0	0
How many $n's$ of type 3	0	0	9274	8880	9918	9987
Corr. Coeff. $r_{n,\sigma}$ of n & $\sigma(p, n, \pm)$	0.2054	0.2892	-0.0091	-0.0813	-0.0947	-0.1501
Corr. Coeff. $r_{n,max}$ of n & $\max(p, n, \pm)$	0.0881	0.1771	0.0711	0.1078	0.0220	0.0029
Mode of $\sigma(p, n, \pm)$ (and its frequency)	52 (190)	39 (220)	10 (23)	33 (19)	17, 16 (8)	NA
Mean of $\sigma(p, n, \pm)$	84.87	56.28	146.41	160.84	66.83	73.41
Standard deviation of $\sigma(p, n, \pm)$	46.59	30.48	82.64	101.25	30.58	39.16

[1] See Appendix 1 for a brief explanation of the JavaScript algorithm, as well as the hyperlink to GitHub folder containing the algorithm itself.
[2] See Appendix 2 for a recall of the definition.
[3] Estimation of Hailstone Radii using the Collatz Conjecture [Data set]. Zenodo. https://doi.org/10.5281/zenodo.7034175

Chart 1: Plot of $\sigma(3, n, +)$ as a function of n



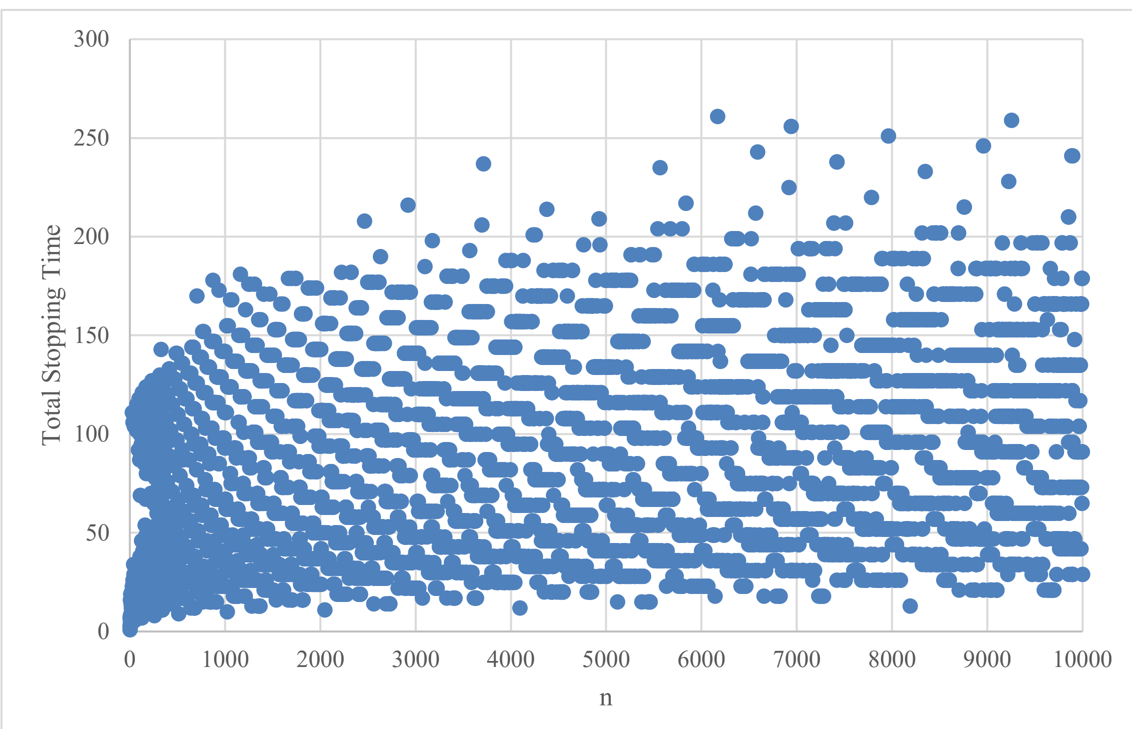


Chart 2: Plot of $\sigma(3, n, -)$ as a function of n

Across the various $(p, n, \pm)$, we tested the hypothesis that the max number or the total stopping time increases as either n becomes larger or p does. However, we found the correlation coefficient to be close to 0 in most cases and no trend either down the list of numbers, or across the values of the prime numbers was found. Hence, it seems that the behavior of the sequence goes randomly up and down, making it hard to solve.

A Real-World Application: Hailstones

How does hail occur?

Let us take a pause from mathematics for a minute and understand the formation of a hailstone, and then we shall come back to formulate the analogy.

A hailstone starts as warm inflow of air is pulled into a storm: the warm air rises and gets caught in an up-draft. When this rises above freezing level (3000-4000 m), water molecules begin to freeze. This forms ice which remains in the cloud. Other water molecules interact with the ice, attach themselves, and a chain reaction occurs that leads to larger ice. When the updraft cannot support the weight anymore, the ice falls for what we know as hail (Thomas, n.d.).

Fig 1. Hail (Thomas, n.d.)

The size of hail is an important consideration in terms of the potential damage it can cause. It can depend on many factors (see analysis later); however, up to 0.5 cm diameter hail happens very frequently and generally does not cause much damage. Warning signs start at the 2 cm size and most damaging hailstones, with highest frequencies are from 4 cm to 8 cm in size (National Weather Service, n.d.).

The largest hailstone ever has even reached 20 cm in size (Allen, et al., 2020). In 2020, more than $20 billion USD in global insurance losses was caused by severe convective storms in USA, with hail as the largest contributor (Allen, et al., 2020). Given hailstones can cause significant damage to property and sometimes even life, it is important and yet very frustrating that our ability to predict the frequency of hailstones and their response to climate variability is limited. This is a bit akin to our ability to predict the maximum value and stopping time for the Collatz sequence.

Advanced data analysis techniques have furthered our ability to assess the Collatz conjecture. Similarly, networks of impact sensors, trained weather observers, data from hail pads, advancements in radar, and satellite technology have significantly improved our understanding of the hailstone phenomenon (Allen, et al., 2020; Kumjian & Lombardo, 2020).

Dry and Wet Growth

Fig 2: Dry and Wet Growth (Thomas, n.d.)

In his paper *The Theory of Hailstone Formation* (1937), Schumann was the first to produce a detailed mathematical theory for hail growth (Schumann, 1938). At that time, the generally accepted theory about the formation of hailstones attributed the process to the capture of super-cooled water drops which lie in the path of the hailstone. More precisely, there was the misconception that only water that can freeze can be accreted (grow), with the surplus part forced to shed. However, in 1959 List discovered spongy ice in hailstones, a form of accretion of super-cooled drops in which the heat transfer is inadequate to freeze all the water, but the excess water is still included within the growing ice. This led to the next theory of hailstone growth (List, 1963), and now the two different types are referred as dry growth (the former) and wet growth (the latter). This is consistent with the change of temperature inside the cloud and the “layered look’’ of hails. The temperature of the region the hail is crossing determines the type of growth: the dry one happens when the air temperature is well below the freezing point, making the droplets instantly freeze, which in turns leaves cloudy ice layer. If the temperature is below freezing (<0° C) but not below -30° C, then the phenomenon of wet growth occurs; this type of growth is slow and lets the air bubbles escape giving the clear ice layer.

The Physics Behind the Formation of Hailstones

To run simulations and make comparisons between the model of a real-life hailstone and a Collatz-like hailstone, we need a treatment of hailstone growth from the physical point of view. Although there are three main shapes of hails (conical, irregular and spheroidal, which can be sphere or ellipsoids) (Macklin, 1977) and while the predominant form of large hailstones is triaxial ellipsoid (List, 1963), we assume the all the hailstones that we consider have spherical symmetry: this not only simplifies the computations (Macklin, 1977, p. 74), but it is also consistent with the assumptions made by most papers on the matter. However, according to some papers (Nelson, 1983, p. 3), this leads to an underestimation of the size of hails.

Because of this assumption, we will also accept that the growth of the hailstone is instantaneously homogeneous, and that the density of the hailstone is constant. This is apparently in contrast with the dry and wet growth (Kumjian & Lombardo, 2020). Nevertheless, despite there being a difference of density between the layers (List,

1963) with spongy ice having higher density, such layers are rare and in general the density approximately ranges from $0.88 g/cm^3$ to $0.917 g/cm^3$ (Macklin, 1977, p. 75). Hence, for computational purposes, the hailstone density may be taken to be equal to $0.9 g/cm^3$.

Homogeneity assumptions are made also about the cloud: hail is associated with high, vertical cumulonimbus clouds, the kind of clouds that produce severe thunderstorms. The cloud base is supposed to be $2km$ and the top $10km$, with hailstone embryos of starting radius of $0.30\ cm$, and its density is considered as constant with a value of $0.5 g/m^3$ (Schumann, 1938, pp. 3-9) and (Nelson, 1983, pp. 1965-1973).

Moreover, no electrical forces are supposed to be involved and the process is viewed as exclusively mechanical. Also, no mass is lost due to melting, motivated also by the fact that hailstones with radius greater than 0.5cm (emphasized in this study) lose little mass due to melting (Gokhale, 1975). Therefore, the forces which the hailstone is subjected to are:

1. Gravity $F_G = gM_H$ (where M_H is the mass of the hailstone);
2. Drag force $F_d = \frac{1}{2} d V_H^2 K_d A$ where d is the density of the fluid (namely, the cloud), V_H is the velocity of the hailstone with respect to the fluid, A is the cross-sectional area - which is in this case πR_H^2, being R_H the radius of a spherical hailstone - and K_d is the drag coefficient – a dimensionless number;
3. The force coming from the total momentum imparted to the water from the hailstone, during the perfectly inelastic collision. Consistently with (Macklin, 1977, p. 66), the velocity of the single water droplets with respect to the cloud is assumed to be zero.

Most measurements of the drag coefficients of hailstones (Nelson, 1983) lead to the conclusion that a reasonable value for the drag coefficient of spherical hailstones is about 0.55, without further assumption about the smoothness of hails; indeed it seems they are unaffected by the *drag crisis* (Macklin, 1977, p. 76).

It has been observed and assumed (Macklin, 1977, p. 66) that hailstones grow only when they drop: therefore the equation of the motion is the one for a generalized variable mass system, where no mass is lost: hence, Newton's second law of motion gives us the change of the momentum with respect to the frame of the cloud as compared with equation (10) and (20) in (Biswal, 2021).

$$\overrightarrow{F_d} + \overrightarrow{F_G} = M_H(t)\frac{d\overrightarrow{V_H}(t)}{dt} - (-\overrightarrow{V_H}(t))\frac{dM_H(t)}{dt}. \quad (6)$$

Taking as the positive sign the downward one, we have

$$-F_d + F_G = M_H(t)\frac{dV_H(t)}{dt} - V_H(t)\frac{dM_H(t)}{dt}. \quad (7)$$

Since we are assuming that our hailstones have spherical shape and constant density, the formula relating the mass of the hailstone M_H to its radius R_H is:

$$M_H = \frac{4}{3}\pi R_H^3 \rho, \quad (8)$$

which implies that the variation is

$$\frac{dM_H}{dt} = 4\pi\rho R_H^2 \frac{dR_H}{dt}. \quad (9)$$

While traveling, the hailstone "scoops" the water droplets that it meets, and in an instant of time dt the amount of water collected is given by the cylinder of height $V_H\, dt$ and of cross-sectional area πR_H^2. Therefore, we have

$$\frac{dM_H}{dt} = 4\pi\rho R_H^2 \frac{dR_H}{dt} = \pi R_H^2 V_H d \quad (10)$$

which implies that the rate of change of the radius in function of the time is

$$\frac{dR_H}{dt} = \frac{V_H d}{4\rho}. \quad (11)$$

However, we need a formula which correlates the change of the radius to the change of height of the hailstone. To get it, consider the total vertical velocity of the hailstone with respect to the ground: on one hand, it can be expressed as dh_H / dt , and on the other hand it is $u - V_H$, where u is the velocity of the cloud w.r.t. the ground (the updraft speed). Thus, using the chain rule we have

$$\frac{dR_H}{dh_H}\frac{dh_H}{dt} = \frac{V_H d}{4\rho} \quad\Rightarrow\quad \frac{dR_H}{dh_H}(u - V_H) = \frac{V_H d}{4\rho} \quad\Rightarrow\quad \frac{dR_H}{dh_H} = \frac{V_H d}{4\rho(u-V_H)}. \qquad (12)$$

Despite we could use a better approximation for the velocity of hailstone V_H, coming from equation *(7)*, we will use the fall-speed derived from the balance between gravity and the drag force for spherical particles:

$$V_H = \left(\frac{8g\rho R_H}{3K_d d}\right)^{\frac{1}{2}}. \qquad (13)$$

Denote by C the constant $\left(\frac{8g\rho}{3K_d d}\right)^{\frac{1}{2}}$ in equation *(13)*, we can rewrite equation *(12)* as

$$\left(\frac{u}{C\sqrt{R_H}} - 1\right) dR_H = \frac{d}{4\rho}\, dh_H. \qquad (14)$$

Then we can integrate the previous ordinary differential equation, obtaining

$$R_{H,fin} = \frac{u}{C} + \left(\left(\frac{u}{C}\right)^2 + R_{H,in} + \frac{d}{4\rho}(h_{in} - h_{fin}) - \frac{2u}{C}\sqrt{R_{H,in}}\right)^{1/2}. \qquad (15)$$

For clarity, we summarise the notation used in the paper:

R_H : radius of the hailstone
h_H : height of the hailstone
h_0 : initial height of the hailstone
V_H : (vertical) speed of the hailstone with respect to the water droplets inside the cloud
M_H : mass of the hailstone
ρ : density of the hailstone
d : density of the cloud
u : updraft velocity of the wind (see Table 11 in Appendix 3 for how the updraft velocity changes with hailstone size)
g : gravity constant (equal to $9.8 m / s^2$)
K_d : drag coefficient

The Analogy with the Conjecture

Given this introduction to hailstones, let us introduce how we want to measure the similarity between the meteorological phenomenon and the mathematical conjecture. We need to compare two frameworks.

Collatz-like hailstones

We assume that the behavior of the trajectory of the hailstone can be related to a Collatz-like function. Rigorously speaking, this means we are assuming that there exists a strictly increasing real-valued function $F: \mathbb{R} \to \mathbb{R}$ such that there exists a sequence of "meaningful times'' $\{t_k\}_{\{k\geq 0\}}$ (where $t_0 = 0$) such that the height $h(t)$ of the hailstone at time $t = t_k > 0$ can be expressed as

$$h(t_k) \coloneqq h_k = F\left(f^{(k)}(\lfloor F^{-1}(h(0))\rfloor)\right) \qquad (16)$$

where the floor function $\lfloor x \rfloor$ of a real number x is the greatest integer less or equal than x. The real functions that we will try in place of F will be square-root, linear, quadratic, logarithmic and exponential like. The sequence of "meaningful times" corresponds to the instances when the motion of the hailstone allegedly changes direction in the process of going up and down.

Real hailstones – Nelson's model

Ideally speaking we could consider the actual trajectories of real hailstones, but we understand them up to a certain extent. We do not have a formula that describes accurately the position $p(t)$ of the hailstone at any time t (notice that $p(t)$ is a 3-dimensional vector $(x(t), y(t), h(t))$ where $(x(t), y(t))$ are the horizontal components). However, we do have some formulae about the growth of the radius of the hailstone (see equation *(15)*), and we have some papers with models and simulated trajectories.

In Nelson's simulation (Nelson, 1983), embryos are distributed uniformly across a subgrid, which lies in vertical between 4km and 10km (included) and subjected to updraft speed varying from 20m/s to 40m/s: of the 1008 embryos 210 eventually grow to diameters ≥ 1.0 cm. The table below (Table 2) shows the number of model grid points that produce hail of a diameter greater than 1.0 as a function of initial height (Nelson, 1983, p. 1973).

Table 2: Nelson's model – Final hailstone diameter (cm) as a function of initial height, h_0

h_0 (km)	Final hailstone diameter (cm)								
	1.0	1.5	2.0	2.5	3.0	3.5	4.0	≥4.5	Total	%
4.0 −5.0	19	-	-	-	-	-	-	-	19	9.1
5.0-6.0	29	8	1	-	-	-	-	-	38	18.1
6.0-7.0	27	11	7	2	1	1	-	-	49	23.3
7.0-8.0	23	9	3	6	1	1	-	1	44	21.0
8.0-9.0	15	9	6	2	-	-	-	1	33	15.7
9.0-10.0	14	8	1	3	-	-	-	1	27	12.9
Total	127	45	18	13	2	2	0	3	210	
Total %	60.5	21.4	8.6	6.2	1.0	1.0	0.0	1.4		

Study Method

We would compare the height $h(t)$ of the real-life hailstones to the one of Collatz-like hailstones but, given the highly unpredictable nature of hailstones and the lack of a math formula, this is not yet doable. Instead, we study what are the effects of the Collatz assumption on the estimated final radii. We compare what we get from our simulations both to real-life final radii measurements, using tables found online (e.g., (Burt, 2020)), and to the values from Nelson's simulation.

We want to stress that a limitation of this study (and in general studies involving hailstones) is that there are very few real observed tabulated data of the final diameter (or radius) of the hailstone in relation to the starting height. However, Table 2 is the ideal tabulation of data, because in Nelson's model we have a table linking the final radii w.r.t. the initial height.

To analyse the suitability of the Collatz conjecture to the observed values of hailstone formation, we used equation *(16)*. For simplicity we will assume that $h_0 = h\,(0)$ is an integer. We will test three types of equations:

a. "Polynomial" $F(h) = (h - h_{\text{inf}})^{\alpha} + 1$, with the exponent α equal to 0.5, 1.0, 2.0 (3 cases)
b. Logarithmic $F(h) = ln(h - h_{\text{inf}}) + 1$ (1 case for log natural, although a quick assessment

showed that $\log_{10}$ does not make a material difference)

c. Exponential $F(h) = exp(h - h_{\mathrm{inf}}) + 1$ (1 case)

As will be clear later, unit of measure is a key variable in equation *(16)*. We shall try units ranging from km to mm, although cm is the default unit (since radius is expressed in cm). All the above would be tried for starting heights ranging from the freezing point, namely 4 km, to 10 km, in increments of 500m to test viability and in increments of 100m for viable solutions.

This gives rise to a maximum of types of cases [5] x [7] UoM (from km to mm), equalling 35 scenarios. Each scenario would be tried for 13 values (4.0 km to 10.0 km in increments of 500m) and for 61 values for viable solutions only (4.0 km to 10.0 km in increments of 100m) depending on the starting height. This would then be compared to the real observed values in nature (Nelson, 1983) and we will use goodness-of-fit to estimate the best possible scenario and equation. We assess, as feasible, only those functions where the following criteria are met:

1) Most (>50%) of the estimated radii values are in the "believable" category for hailstone radii i.e., less than 10 cm;
2) There is reasonable dispersion in the range of estimated radii to analyze goodness-of-fit (some values between 0.5 cm and 2.5 cm and range should be at least 1.0 cm;
3) The estimated radii should have enough points of inflection (i.e., rises and drops like a Collatz Conjecture or observed hailstone radii) and not be a monotonic function. For instance, if the estimated values of R_h for different values of h, steadily increase (or decrease) as h increases (or decreases), then this would not be analogous to how the values of $f(n)$ fluctuate (which increase and decrease multiple times before converging to 1) for different values of n. Hence, we would not consider such a function to behave analogously to a Collatz function and would not consider them as feasible

Table 3: Summarising the assumptions for values for the variables used in the computation

Variable	Definition	Unit	Value
ρ	Hailstone density	g/ cm^3	0.9
g	Gravity constant	cm /s^2	980
R_0	Initial radius	cm	0.30
K_d	Drag coefficient	no unit	0.55
d	Density of cloud	g/ cm^3	0.0000005
u	updraft speed	m /s	25
h_{inf}	Bottom height	cm	200000
C	Constant	$m^{1/2}$/s	924.80

Analysis

If we consider the sequence of heights $\{h_k\}_{k\geq 0}$ corresponding to the Collatz sequence via the conversion formula, then

- if $h_{k+1} \geq h_k$ (hailstone rises in height) then there is no change in the radius. In particular $R_{k+1} = R_k$;
- if $h_{k+1} \leq h_k$ (hailstone drops in height) then the radius changes according to equation 15. In particular $R_{k+1} > R_k$.

Then the final radius R_{fin} is achieved when h_{fin} is equal to 2000 meters, which corresponds when $F(h_{fin}) = 1$ that is when the Collatz sequence reaches 1. We computed the final radii for given starting heights h_0 and the conversion formulas above reported, and we summarize the results in the tables that follow. The first column contains the initial height (the height of the embryo of the hailstone), while the others contain the final radii expressed in cm. For the initial height, we consider values from 4 km (or equivalent, depending on the unit of measure) to 10 km (or equivalent). This is because our observed comparison data is Table 2 (Nelson, 1983), where the h_0 ranges from 4.0 km to 10.0 km (extreme included).

We changed the updraft speed from 25 m/s to other values between 20 m/s to 50 m/s but there was no material difference in the estimation of the radii. For instance, simulating the radii for the linear equation with α = 1.0 and unit of measure being cm, the change was less than 0.01 cm; it was found to be similar for other scenarios too. Hence, a constant updraft speed of 25 m/s was assumed.

Scenario 1: Polynomial equation for α = 0.5

Table 4: Values of the final radius (cm) with α = 0.5[4]

<table>
<tr><td>h_0 (m)</td><td>Km</td><td>Hm</td><td>Dm</td><td>m</td><td>dm</td><td>cm</td><td>mm</td></tr>
<tr><td>4000</td><td>0.30</td><td>0.30</td><td>0.57</td><td>9.35</td><td colspan="3" rowspan="13">All values getting increasingly larger and significantly greater than 10.0 cm (typically greater than 10000 cm) as unit of measure gets smaller; hence not assessed</td></tr>
<tr><td>4500</td><td>0.30</td><td>0.30</td><td>31.71</td><td>8.49</td></tr>
<tr><td>5000</td><td>0.30</td><td>32.83</td><td>0.44</td><td>40.79</td></tr>
<tr><td>5500</td><td>0.30</td><td>0.30</td><td>38.01</td><td>20.81</td></tr>
<tr><td>6000</td><td>0.30</td><td>32.83</td><td>0.61</td><td>59.62</td></tr>
<tr><td>6500</td><td>0.30</td><td>0.31</td><td>6.79</td><td>26.18</td></tr>
<tr><td>7000</td><td>0.30</td><td>0.31</td><td>33.14</td><td>1,032.65</td></tr>
<tr><td>7500</td><td>0.30</td><td>0.30</td><td>6.46</td><td>132.50</td></tr>
<tr><td>8000</td><td>0.30</td><td>0.31</td><td>7.21</td><td>68.55</td></tr>
<tr><td>8500</td><td>0.30</td><td>0.30</td><td>35.19</td><td>92.36</td></tr>
<tr><td>9000</td><td>0.30</td><td>32.82</td><td>33.21</td><td>164.32</td></tr>
<tr><td>9500</td><td>0.30</td><td>0.30</td><td>277.36</td><td>32,162.28</td></tr>
<tr><td>10000</td><td>0.30</td><td>0.32</td><td>7.97</td><td>143.77</td></tr>
</table>

Considering the results, we can see that no case respects all the viability criteria. Indeed, for Km, there is virtually no change in the radius (0.30 cm) and hence no variation that we can use goodness-of-fit to assess against Table 2. For Hm, of the 13 values assessed, 10 of them showed virtually no variation (between 0.30 and 0.31 cm) and 3 values that did show a variation, were very large (>32 cm) to be considered believable. For Dm, there were only 3 values (out of 13 estimated) that were all in a narrow range (<1.0 cm), 4 values that were large and yet believable (5.0 cm < r < 10.0 cm), and 7 values that were very large (>10.0 cm). For m, only 2 values were <10.0 cm and 11 values unbelievably large (>10.0 cm). For dm, cm, mm, all values were greater that 10.0 cm.

Hence, as we can see, Scenario 1 did not yield any feasible options that meet the three criteria: believability of size, reasonable dispersion, sufficient points of inflection.

Scenario 2: Polynomial equation for α = 1.0

Table 5: Values of the final radius (cm) with α = 1.0

h_0(m)	Km	Hm	Dm	m	dm	cm	mm
4000	0.30	0.30	0.30	0.30	0.35	0.48	11.54

[4] Estimation of Hailstone Radii using the Collatz Conjecture [Data set]. Zenodo. https://doi.org/10.5281/zenodo.7034175

4500	0.30	0.30	0.30	0.30	0.50	1.84	4.86
5000	0.30	0.31	0.30	0.30	0.35	1.43	3.35
5500	0.30	0.30	0.31	0.30	0.36	0.62	21.17
6000	0.30	0.31	0.30	0.31	0.36	1.56	4.62
6500	0.30	0.30	0.30	0.30	0.47	0.70	5.99
7000	0.30	0.30	0.31	0.34	0.45	1.52	4.63
7500	0.30	0.30	0.30	0.31	0.43	0.66	12.65
8000	0.30	0.30	0.30	0.30	0.49	0.90	7.89
8500	0.30	0.30	0.31	0.31	0.61	0.88	16.31
9000	0.30	0.31	0.31	0.31	0.35	1.34	31.99
9500	0.30	0.30	0.32	0.56	0.45	1.51	60.02
10000	0.30	0.30	0.30	0.31	0.52	1.15	10.41

Considering the results, we can see that only one case respects all the viability criteria. Indeed, for Km, Hm, Dm, m, there is virtually no change in the radius (all values between 0.30 cm and 0.34 cm) and hence no variation that we can use goodness-of-fit to assess against Table 2. For dm, too, all values are <1.0 cm and again, the range is very narrow. For cm, we notice that all the values are believable (<10.0 cm), there is a good range of values between 0.48 cm and 1.84 cm, and the values are not monotonic; they tend to increase and decrease with 9 points of inflection. For mm, only 6 values were believable (<10.0 cm) while 7 values were very large (>10.0 cm)

Hence, as we can see, Scenario 2 yields one feasible option (unit of measure as cm) that meets the three criteria of: believability of size, reasonable dispersion, sufficient points of inflection.

Scenario 3: Polynomial equation for $\alpha = 2.0$

Table 6: Values of the final radius (cm) with $\alpha = 2.0$

h_0 (m)	Km	Hm	Dm	m	dm	cm	mm
4000	0.30	0.30	0.30	0.30	0.30	0.33	0.59
4500	0.30	0.30	0.30	0.30	0.30	0.34	0.66
5000	0.30	0.30	0.30	0.30	0.30	0.34	0.73
5500	0.30	0.30	0.30	0.30	0.31	0.35	0.81
6000	0.30	0.30	0.30	0.30	0.31	0.36	0.88
6500	0.30	0.30	0.30	0.30	0.31	0.37	0.95
7000	0.30	0.30	0.30	0.30	0.31	0.37	1.02
7500	0.30	0.30	0.30	0.30	0.31	0.38	1.09
8000	0.30	0.30	0.30	0.30	0.31	0.39	1.16
8500	0.30	0.30	0.30	0.30	0.31	0.39	1.24
9000	0.30	0.30	0.30	0.30	0.31	0.40	1.31
9500	0.30	0.30	0.30	0.30	0.31	0.41	1.38
10000	0.30	0.30	0.30	0.30	0.31	0.42	1.45

Considering the results, we can see that no case respects all the viability criteria. Indeed, for Km, Hm, Dm, m, dm, cm there is virtually no change in the radius (all values between 0.30 cm and 0.42 cm) and hence no variation that we can use goodness-of-fit to assess against Table 2. For mm, the values are all believable (<1.5 cm) and there is some variation (ranging from 0.59 cm to 1.45 cm). However, the values monotonically increase with height unlike how the Collatz function or observed formation of hail in a hailstone behave. Hence, we do not consider this option as feasible either.

Hence, as we can see, Scenario 3 yields no feasible option that meets the three criteria: believability of size, reasonable dispersion, sufficient points of inflection.

Scenario 4: Natural logarithm (ln) equation

Table 7: Values of the final radius (cm) with ln-equation

h_0 (m)	Km	Hm	Dm	m	dm	cm	mm
4000	0.77	3.18E+20	These cases are not explicitly reported since the final value exceeds 10.0 cm; in fact, most values exceed 1.0E+10 cm				
4500	0.77	1.20E+10					
5000	0.30	NA					
5500	0.30	1.96E+14					
6000	0.77	NA					
6500	0.77	1.57E+62					
7000	0.77	2.91E+93					
7500	0.77	1.96E+15					
8000	1.90E+14	4.16E+72					
8500	1.90E+14	1.37E+36					
9000	0.30	NA					
9500	0.30	8.43E+30					
10000	1.96E+14	4.75E+98					

Considering the results, we can see that no case respects all the viability criteria. For Km, we have 10 values that are within a very narrow range (between 0.3 and 0.8 cm) and 3 values that are unbelievably large (> 1.9E+14). For Hm, 10 values are unbelievably large (>1.2E+10) while 3 values are so large that they could not be computed in Excel. For all other units of measure, the values are larger than 1.0E+10 or could not be computed. Hence, as we can see, Scenario 4 yields no feasible option primarily because the values are unbelievably large.

Scenario 5: Exponential equation

Table 8: Values of the final radius (cm) with exp-equation

h_0 (m)	Km	Hm	Dm	m	dm	cm	mm
4000	0.30	0.30	0.30	0.30	0.30	0.33	0.59
4500	0.30	0.30	0.30	0.30	0.30	0.34	0.66
5000	0.30	0.30	0.30	0.30	0.30	0.34	0.73
5500	0.30	0.30	0.30	0.30	0.31	0.35	0.81
6000	0.30	0.30	0.30	0.30	0.31	0.36	0.88
6500	0.30	0.30	0.30	0.30	0.31	0.37	0.95
7000	0.30	0.30	0.30	0.30	0.31	0.37	1.02
7500	0.30	0.30	0.30	0.30	0.31	0.38	1.09
8000	0.30	0.30	0.30	0.30	0.31	0.39	1.16
8500	0.30	0.30	0.30	0.30	0.31	0.39	1.23
9000	0.30	0.30	0.30	0.30	0.31	0.40	1.30
9500	0.30	0.30	0.30	0.30	0.31	0.41	1.38

10000	0.30	0.30	0.30	0.30	0.31	0.42	1.45

Considering the results, we can see that no case respects all the viability criteria. Indeed, for Km, Hm, Dm, m, dm, cm there is virtually no change in the radius (all values between 0.30 cm and 0.42 cm) and hence no variation that we can use goodness-of-fit to assess against Table 2. For mm, the values are all believable (<1.5 cm) and there is some variation (ranging from 0.59 cm to 1.45 cm). However, the values monotonically increase with height unlike how the Collatz function or observed formation of hail in a hailstone behave. Hence, we do not consider this option as feasible either.

Hence, as we can see, Scenario 5 yields no feasible option that meets the three criteria of: believability of size, reasonable dispersion, sufficient points of inflection.

Therefore, summarizing our analysis across the 5 scenarios assesses, the only viable solution that meets all three criteria is the linear equation ($\alpha = 1$) and with centimeters as unit of measure.

Linear case in cm: the $\chi^2 - tests$

Now we proceed to address the second question in this case: how suitable the description is. In order to do so we compare our values to Nelson's model data. To assess the suitability of the function using goodness-of-fit, we further estimated the radii in increments of 100m (to keep the uniform distribution assumption from 4km to 10km, such as the one used by Nelson) and obtained the values in Table 9 below.

Table 9: Estimated radius (and diameter) in cm for different heights (in km) and increments of 100m each

Height (km)	Radius (cm)	Diameter (cm)		Height (km)	Radius (cm)	Diameter (cm)		Height (km)	Radius (cm)	Diameter (cm)
4.0	0.48	0.96		5.1	0.53	1.07		6.1	0.64	1.28
4.1	0.65	1.29		5.2	0.70	1.41		6.2	20.39	40.79
4.2	3.36	6.71		5.3	0.59	1.18		6.3	0.60	1.21
4.3	0.46	0.91		5.4	0.92	1.83		6.4	0.73	1.47
4.4	0.60	1.20		5.5	0.62	1.24		6.5	0.70	1.41
4.5	1.84	3.68		5.6	0.64	1.27		6.6	108.78	217.57
4.6	0.64	1.28		5.7	34.20	68.39		6.7	0.67	1.33
4.7	0.52	1.04		5.8	1.23	2.46		6.8	0.79	1.58
4.8	0.74	1.47		5.9	0.60	1.20		6.9	0.68	1.36
4.9	1.02	2.04		6.0	1.56	3.12		7.0	1.52	3.03
5.0	1.43	2.86								

Height (km)	Radius (cm)	Diameter (cm)		Height (km)	Radius (cm)	Diameter (cm)		Height (km)	Radius (cm)	Diameter (cm)
7.1	0.91	1.82		8.1	0.94	1.89		9.1	0.79	1.57
7.2	0.77	1.53		8.2	1.77	3.53		9.2	1.51	3.03
7.3	2.09	4.17		8.3	0.86	1.72		9.3	1.05	2.10
7.4	1.29	2.57		8.4	1.00	1.99		9.4	1.80	3.61
7.5	0.66	1.32		8.5	0.88	1.76		9.5	1.51	3.01
7.6	20.57	41.14		8.6	1.71	3.43		9.6	1.39	2.77
7.7	1.14	2.28		8.7	0.86	1.72		9.7	1.06	2.11
7.8	1.37	2.73		8.8	1.13	2.25		9.8	6.10	12.21

7.9	1.22	2.43	8.9	1.40	2.80	9.9	1.78	3.55
8.0	0.90	1.79	9.0	1.34	2.67	10.0	1.15	2.31

We summarise the values in Table 10 by tabulating the rows in km, in increments of 1.0 km and grouping diameters for (almost) 1.0 cm (greater than 0.91), 1.0-1.49 cm, 1.5-1.99 cm, 2.0-2.49 cm, 2.5-2.99 cm, 3.0-3.49 cm, 3.5-3.99 cm, 4.0-4.49 cm, 4.5 cm and above.

Table 10: Scenario 2 model - final hailstone diameter (cm) as a function of h_0

h_0 (km)	Final hailstone diameter (cm)									
	1.0	1.5	2.0	2.5	3.0	3.5	4.0	≥4.5	Total	%
4.0-5.0	2	5	-	1	1	-	1	1	11	18.0%
5.0-6.0	-	6	1	1	-	1	-	1	10	16.4%
6.0-7.0	-	6	1	-	-	1	-	2	10	16.4%
7.0-8.0	-	1	3	2	2	-	-	2	10	16.4%
8.0-9.0	-	-	5	1	2	1	1	-	10	16.4%
9.0-10.0	-	-	1	3	1	2	2	1	10	16.4%
Total	2	18	11	8	6	5	4	7	61	
Total %	3.3%	29.5%	18.0%	13.1%	9.8%	8.2%	6.6%	11.5%		

We want to compare Table 10 to Table 2 (Nelson, 1983) and estimate feasibility of our hypothesis using $\chi^2 - test$ on the frequency of final hailstone diameters.

Since Nelson's paper focuses on large hailstone, we do the same, checking the hypothesis that Collatz-like description is suitable among the hailstones that exceed 1.0cm as diameter. First, we compare the distribution of the final radii. We consider as possible outcomes all the pairs (h_0, R_{fin}) of Table 10, whose corresponding entry is not null: thus, there are 31 possible outcomes. Hence the degrees of freedom are 31-1=30 and with a significance level of 0.05, the critical value for the test is 43.773 (see (Turney, 2022)). After normalization, the distribution frequencies of the expected (Collatz) and the observed (Nelson) are reported in the following tables.

Observed (normalized)	Final hailstone diameter (cm)							
h_0 (km)	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
4.0-5.0	9.05%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
5.0-6.0	13.81%	3.81%	0.48%	0.00%	0.00%	0.00%	0.00%	0.00%
6.0-7.0	12.86%	5.24%	3.33%	0.95%	0.48%	0.48%	0.00%	0.00%
7.0-8.0	10.95%	4.29%	1.43%	2.86%	0.48%	0.48%	0.00%	0.48%
8.0-9.0	7.14%	4.29%	2.86%	0.95%	0.00%	0.00%	0.00%	0.48%
9.0-10.0	6.67%	3.81%	0.48%	1.43%	0.00%	0.00%	0.00%	0.48%

Expected (normalized)	Final hailstone diameter (cm)							
h_0 (km)	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
4.0-5.0	3.28%	8.20%	0.00%	1.64%	1.64%	0.00%	1.64%	1.64%
5.0-6.0	0.00%	9.84%	1.64%	1.64%	0.00%	1.64%	0.00%	1.64%
6.0-7.0	0.00%	9.84%	1.64%	0.00%	0.00%	1.64%	0.00%	3.28%
7.0-8.0	0.00%	1.64%	4.92%	3.28%	3.28%	0.00%	0.00%	3.28%
8.0-9.0	0.00%	0.00%	8.20%	1.64%	3.28%	1.64%	1.64%	0.00%
9.0-10.0	0.00%	0.00%	1.64%	4.92%	1.64%	3.28%	3.28%	1.64%

Therefore the χ^2value is computed as follows:

$(O-E)^2/E$	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	Total
4.0-5.0	10.150	8.20	-	1.639	1.639	-	1.639	1.639	24.905
5.0-6.0	-	3.692	0.825	1.639	-	1.639	-	1.639	9.436
6.0-7.0	-	2.149	1.750	-	-	0.825	-	3.279	8.004
7.0-8.0	-	4.272	2.476	0.054	2.395	-	-	2.395	11.593
8.0-9.0	-	-	3.478	0.288	3.279	1.639	1.639	-	10.324
9.0-10.0	-	-	0.825	2.476	1.639	3.279	3.279	0.825	12.323
Total	10.150	18.311	9.355	6.097	8.953	7.3835	6.557	9.778	76.584

Since the original sample was made of 210 hailstones, we need to multiply the result by 210/100=2.1, obtaining that the χ^2-value is 160.826. Now we want to do further comparisons but considering only particular ranges. We start with hailstones whose diameter exceeds 2.0cm. Since the observed sample is made of 38 hailstones (210-127-45=38) and the expected sample consists of 41 hailstones, we normalize the values by 38/41, obtaining Table 13. In this case the critical value is 11.1 (5 degrees of freedom and the same significance level of 0.05), and the χ^2-value is 19.53.

Table 13: Comparison of *big* final hailstone diameter (cm) - critical value 11.1

	2.0	2.5	3.0	3.5	4.0	≥4.5	Total
Collatz-like	10.2	7.4	5.56	4.6	3.7	6.5	~38
Nelson's model	18	13	2	2	-	3	38
$(O-E)^2/E$	5.96	4.24	2.28	1.47	3.7	1.88	19.53

Besides the under bound about the final radius, let now us fix some stricter ranges of initial heights (h_0:7.0-8.0; h_0:8.0-9.0; h_0:9.0-10.0) and do the same χ^2-tests.

Table 14: Comparison with fixed h_0:7.0-8.0 and diameter $\geq 2cm$ - critical value 7.8

h_0:7.0-8.0	Final hailstone diameter (cm)								
	1.0	1.5	2.0	2.5	3.0	3.5	4.0	≥4.5	Total
Collatz-like	x	x	4	2.7	2.7	-	-	2.7	12
Nelson's model	x	x	3	6	1	1	-	1	12
$(O-E)^2/E$	x	x	0.25	6.33	1.07	x	x	1.07	8.72

Table 15: Comparison with fixed h_0:8.0-9.0 and diameter $\geq 2cm$ - critical value 9.5

h_0:8.0-9.0	Final hailstone diameter (cm)								
	1.0	1.5	2.0	2.5	3.0	3.5	4.0	≥4.5	Total
Collatz-like	x	x	4.5	0.9	1.8	0.9	0.9	-	9
Nelson's model	x	x	6	2	-	-	-	1	9
$(O-E)^2/E$	x	x	0.5	1.3	1.8	0.9	0.9	x	5.4

Table 16: Comparison with fixed h_0:9.0-10.0 and diameter $\geq 2cm$ - critical value 11.1

h_0:9.0-10.0	Final hailstone diameter (cm)								
	1.0	1.5	2.0	2.5	3.0	3.5	4.0	≥4.5	Total
Collatz-like	x	x	0.5	1.5	0.5	1	1	0.5	5
Nelson's model	x	x	1	3	-	-	-	1	5
$(O - E)^2/E$	x	x	0.5	1.5	0.5	1	1	0.5	5

Conclusion

All our tests indicate that the analogy between the Collatz sequences and hailstones is quite poor. Both our viable criteria based on real-life observations and χ^2-tests tell us that our initial hypotheses should be rejected: indeed, in almost all cases the χ^2-value is above the critical value of the test. If we focus only on big radii and we restrict the initial height, the χ^2-value is under the critical value: however, this phenomenon is explained by the fact that the size of the sample abruptly decreases when we focus on those extreme cases and therefore, we need more data to have precise conclusions in those scenarios.

Limitations

The paper has certain limitations, in terms of assumptions made and lends itself to some natural next steps that would worth evaluating. The analysis could be conducted for different geometrical shapes of a hailstone, other than the spherical shape assumed in this paper, e.g., triaxial ellipsoid. We discussed the generalised Collatz function being $pn \pm 1$ (for odd n), whence we simulated the hailstone formation only with the equation $3n + 1$ (for odd n). The same procedure could be done for other values of p and sign $(+, -)$. Appendix 3 shows how the updraft speed varies with the size of the hailstone. We did a quick simulation of different updraft speeds but then used a constant (25 m/s) for our paper. A more detailed study, linking size to different values of the updraft speed (from 20 m/s to 50 m/s) would be another useful addition lending greater rigour to the analysis. For the generalised equation *(16)*, we tried five forms of equations – polynomial with $\alpha = 0.5$, 1.0 and 2.0, logarithmic and exponential. It would be a useful to try other forms of equation (e.g., other values of α, especially from 0.9 to 1.1). Finally, one of the biggest limitations of the paper is the fact that we have used only one table of "observed" data with 210 values and having a non-normal distribution inherently. While many papers refer to several collated values of hailstone data, but our research did not reveal any other well documented tables linking radii (or mass) to the height; it would be useful to reach out to meteorological societies and obtain more such data to do the goodness-of-fit test with.

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Appendices

Appendix 1: Java Code

Here we add a brief discussion about the algorithm used to test the Collatz-like functions: for the algorithm itself, see GitHub (link). The program takes 4 inputs and generates 4 outputs.

The inputs are the following:

- Range of numbers you want to test the program for $(n1, n2)$
- Prime number (p)
- Sign $(+$ or $-)$

For each number $n_1 \leq n \leq n_2$, and for the Collatz function f ($n/2$ if even or $pn \pm 1$ if odd), it executes f and saves the resultant value in an array. It checks if the resultant value is either 1 or has already been saved in the array earlier (in which case the array would start duplicating). If either of the conditions are met, the program exits the loop and produces the outputs; else it keeps iterating function f.

There is a limitation that if the resultant number reaches or exceeds 2147483647, then we conclude that there is no convergence or duplication; this is a Java programming limitation.

Finally, the algorithm produces as outputs:

- Number (n) for which the calculation is done
- The maximum number it reaches in the successive iterations of f
- The total stopping time to reach 1 or to a duplicate number
- The converging number (either 1, or the duplicate number, or "no convergence" should the formula exceed 2147483647).

Appendix 2: Definition of Pearson correlation coefficient (r_{xy}) and Chi-square (χ^2)

Correlation coefficient (Glen, n.d.) is used to determine how strong a relationship is between data. It is a value between -1 and +1, where:

- +1 indicates a strong positive relationship
- -1 indicates a strong negative relationship
- 0 indicates no relationship at all.

The most common measure of correlation in statistics is the Pearson Product Moment Correlation (PPMC) or simply known as the Pearson correlation coefficient. It shows the linear relationship between two sets of data $\{x_i\}$ and $\{y_i\}$. It is expressed by the formula

$$r_{xy} = \frac{n\sum_{i=1}^{n} x_i\, y_i - (\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i)}{[n\sum_{i=1}^{n} x_i{}^2 - (\sum_{i=1}^{n} x_i)\hat{}2\ ]^{1/2}[n\sum_{i=1}^{n} y_i{}^2 - (\sum_{i=1}^{n} y_i)\hat{}2\ ]^{1/2}}. \quad (17)$$

A chi-square statistic (Glen, n.d.) is a way to show a relationship between two variables. It is a single number that tells you how much difference exists between your observed counts and the counts you would expect if there were no relationship at all in the population. A low value for chi-square means there is a high correlation between the two sets of data. It can be calculated using the formula

$$\chi^2 = \sum_i \frac{(O_i - E_i)^2}{E_i} \qquad (18)$$

where O_i is the observed frequency and E_i is the expected one.

In order to conclude whether a chi-square value is low or high, you need to specify the critical value. Common critical values are computed w.r.t. significance level 0.05 (see (Turney, 2022)).

Appendix 3: Relationship between hailstone size and updraft speed

(National Weather Service, n.d., p. 3) provides us with a table linking the size of the hailstone and an approximation of the updraft speed needed to sustain such weight.

Table 11: Hailstone size at different updraft speeds

Hailstone size	Measurement in cm	Updraft speed in m/s
Bb	<0.64	<39
Pea	0.64	39
Marble	1.3	56
Dime	1.8	61
Penny	1.9	64
Nickel	2.2	74
Quarter	2.5	79
Half dollar	3.2	87
Walnut	3.8	97
Golf ball	4.4	103
Hen egg	5.1	111
Tennis ball	6.4	124
Baseball	7.0	130
Teacup	7.6	135
Grapefruit	10.1	158
Softball	11.4	166