HIGH SCHOOL EDITION
Journal of Student Research

Volume 11 Issue 3 (2022)

Complex Roots of Quadratics with the Floor Function

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ABSTRACT

Quadratics are generally a rather well-understood portion of mathematics, with methods of solving quadratics being presented early in one's mathematical development. However, when floor functions, also a well-understood function in mathematics, are inserted into such quadratics, the complexity of such expressions quickly increases. In this paper we extend the work of a previous paper focusing on the behavior of quadratics with the floor function in them by extending our analysis to the complex plane. In our paper, we utilize both algebraic analysis and domain coloring to examine the roots of such functions, and the impact the coefficients of a quadratic have on the roots of such a function. Our paper finds that quadratics with a floor function inside can be solved through finding the intersection of two graphs or through the use of domain coloring. We also show that there is no upper bound on the number of complex roots a single function of the form $z^2 + b\lfloor z \rfloor + c$ can have. Lastly, we examine the behavior of the roots of a quadratic with a floor function inside when the coefficients of the equation are incremented and find similarities to the behavior of a normal quadratic.

Introduction

Quadratic functions, polynomials of the form $x^2 + bx + c$ with $b, c \in \mathbb{R}$, have been extensively studied, their real and complex roots easily derived from the quadratic formula. The floor function, also known as the greatest integer function and denoted as $\lfloor x \rfloor$, is the mathematical function which outputs the greatest integer less than or equal to the input[1]. Most basic equations involving the floor function can be easily solved through inequalities. Though these two types of functions separately have been intricately examined, the combination of the two, which we shall refer to as floored quadratics for the sake of brevity, remain largely unexplored. This paper builds off an existing article exploring quadratics with floor functions in them[2]. The previous paper limited the scope of its investigation to real numbers, thus reducing the complexity of the math for solving for solutions. This paper seeks to expand the scope of the study to complex roots, consequently encompassing all possible roots (both real and complex). To accomplish this, we must first define what the floor of a complex number is. There are multiple ways the floor of a complex number could be defined, but for the purpose of this paper we will use the definition $\lfloor a + bi \rfloor = \lfloor a \rfloor + \lfloor b \rfloor i$.

The primary goal of this research paper is to study the roots for a certain form of floored quadratic. This includes both finding the roots and finding the maximum and minimum possible number of roots that a quadratic of that form can have.

The secondary goal will be to explore the patterns that these roots form on the complex plane when incrementing b or c. In normal quadratics, this behavior is predictable, but in floored quadratics, the behavior is more erratic and difficult to predict.

With the aforementioned definition of the floor of a complex number, we will now define what a floored quadratic is. Since there are three total terms with an x in a quadratic (2 in the x^2 and one in the bx portion of a quadratic polynomial), there are 7 total floored quadratic equations, which are listed below:

- $\lfloor z\rfloor^2 + b\lfloor z\rfloor + c$
- $\lfloor z\rfloor^2 + bz + c$
- $\lfloor z^2\rfloor + b\lfloor z\rfloor + c$
- $\lfloor z^2\rfloor + bz + c$
- $\lfloor z\rfloor z + b\lfloor z\rfloor + c$
- $\lfloor z\rfloor z + bz + c$
- $z^2 + b\lfloor z\rfloor + c$

For simplicity, we will focus on finding the real and complex roots for $z^2 + b\lfloor z\rfloor + c$. We hope that our analysis of this specific equation can help lay the groundwork for analysis of the other 6.

Algebraic Analysis

The first approach taken was a pure mathematical/algebraic method, attempting to simplify the solution down in such a way that a pattern would be yielded to derive complex and real solutions given any b and c. This was done by substituting in $x + yi$ for z and then expanding, collecting like terms, and setting the real part of the quadratic and the imaginary part both equal to 0. This process is shown below.

$$(x + yi)^2 + b\lfloor(x + yi)\rfloor + c = 0$$
$$x^2 + 2xyi - y^2 + b\lfloor x\rfloor + b\lfloor y\rfloor i + c = 0$$

Condensing real and imaginary parts gives us the following:

$$2xyi + b\lfloor y\rfloor i = 0 \rightarrow 2xy + b\lfloor y\rfloor = 0 \qquad (1)$$
$$x^2 - y^2 + b\lfloor x\rfloor + c = 0 \qquad (2)$$

These two equations were powerful in that they allowed us to find the root of any floored quadratic by simply plugging in values of b and c and graphing both graphs. Doing so yielded the roots as intersections. An example of a graph is shown below.

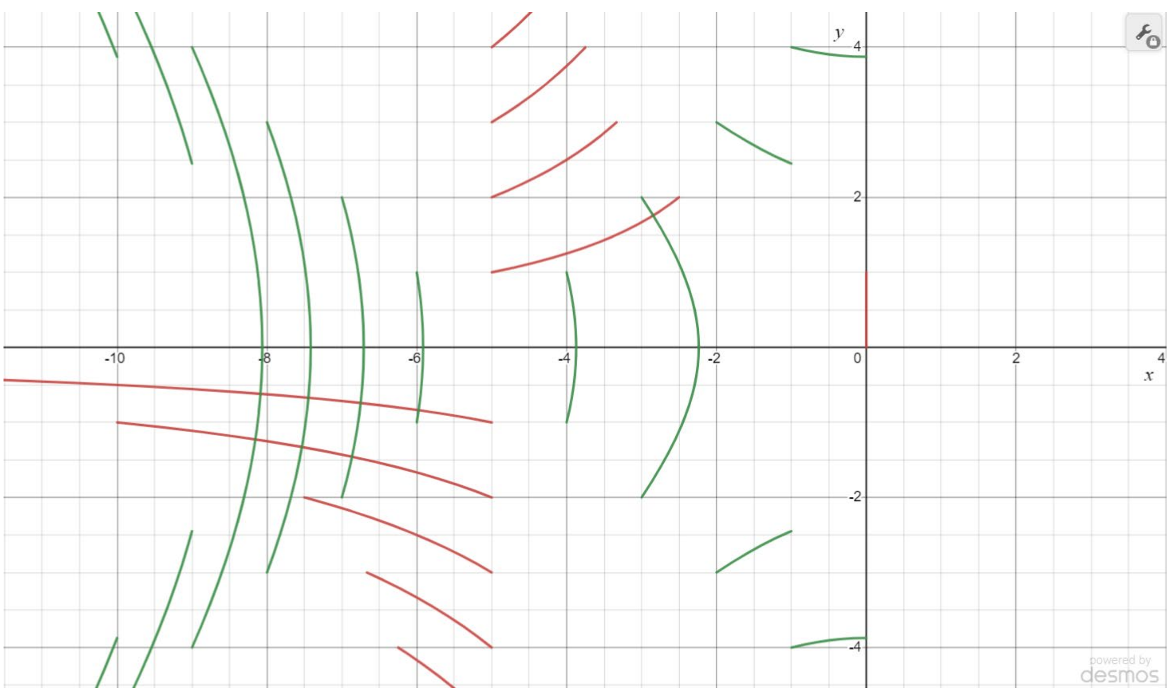


Figure 1. Example of equations (1) (in red) and (2) (in green) plotted together. These equations were derived from the floored quadratic $z^2 + 10\lfloor z\rfloor + 25$, so we used $b = 10$ and $c = 25$.

However, in terms of analyzing the patterns, the algebraic method is insufficient. Though the imaginary and real graphs can be analyzed separately, the intersection points do not display any obvious patterns. Therefore, following the algebraic approach, the quickest method to obtain the solutions would be to create a program to parse through different values for b and c and record the intersection points of the resulting graphs. This method initially seems appealing. Unfortunately, since programs like Desmos do not provide the intersection points in an easily accessible manner, one would have to manually change the values of b and c and record the intersection points. As it is imperative that a vast amount of intersection/solution points be collected such that a graph could be generated for the roots, this method was not viable because of its tediousness.

Although algebra is not a viable strategy for finding patterns in the roots of floored quadratics, it is useful in determining the maximum possible number of roots for a floored quadratic. Although both equations form hyperbolas, if we inspect y in intervals $[d, d+1)$ for integer d, then equation (1) could be interpreted as many small sections of different $\frac{1}{x}$ hyperbolas. Similarly, if we inspect x in intervals $[e, e+1)$ for integer e, equation (2) can be interpreted as many small sections of different $x^2 - y^2$ hyperbolas, almost like they are forming different "layers." These "layers" are disjoint because of the floor function within the equations, and each "layer" is distinguished by the value of $\lfloor y \rfloor$ for equation (1) and $\lfloor x \rfloor$ for equation (2). We define a "layer" of the graph of equation (1) as the portion of the graph of equation (1) that has y-value between two consecutive integer y-values, and the layers of equation (2) will be defined as the same just with consecutive integer x-values. The different "layers" of the graphs of the equations are shown in Figure 2.

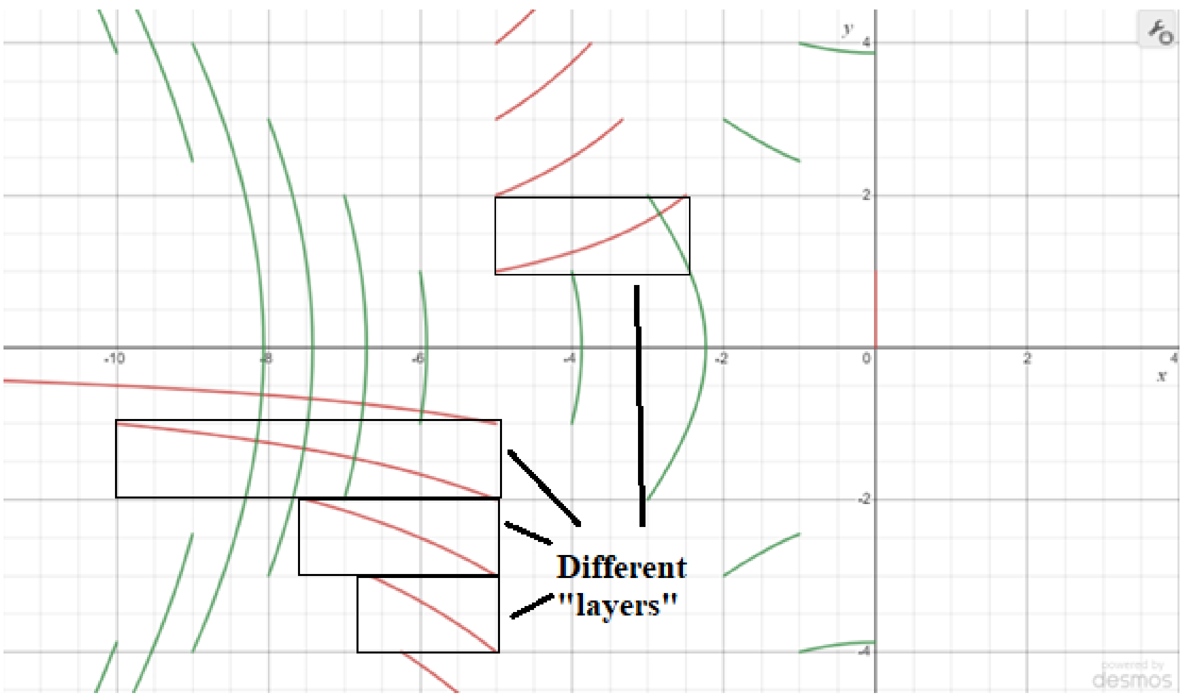


Figure 2. Different "layers" labeled from the graphs shown in Figure 1.

For equation (1), we noticed that as we increased the absolute value of b, the length of the $\frac{1}{x}$ curves increased, except for the curve between -1 and 0. The proof for this is as follows.

Let us fix the value of the floor of y as some integer value d that is not -1 or 0. The range of y for when this is valid is $[d, d+1)$. When solving for x from equation (1), we get that $x = -\frac{b\lfloor y \rfloor}{2y}$. We can then plug in the values of d and $d+1$ for y, as well as plugging in d for $\lfloor y \rfloor$ to get the bounds for x. We can take the absolute value of the difference of these two to get the length of a given "layer." $-\frac{bd}{2d+2}$ is the value of x when y is maximized, and $-\frac{b}{2}$ is the value of x when y is minimized. Taking the absolute value of the difference, we get the expression for the length of a given "layer".

$$\left|-\frac{bd}{2d+2}-\left(-\frac{b}{2}\right)\right|$$
$$=\left|-\frac{bd}{2d+2}+\frac{(b)(d+1)}{2d+2}\right| = |-\frac{bd}{2d+2}+\frac{bd+b}{2d+2}|$$
$$=\left|\frac{b}{2d+2}\right| \qquad (3)$$

This logic does not work for the curve between -1 and 0 nor the curve between 0 and 1 because we must divide by d or $d+1$ for our expression for x, and one of those would be zero in each case. This would give us an undefined value for the length of that layer. For the layer between $y=-1$ and $y=0$, the length will always be infinite. From earlier, we had that $x=-\frac{b\lfloor y\rfloor}{2y}$. Since we are working in the range for y of $[-1,0)$, we know that $\lfloor y\rfloor=-1$. Therefore, for this layer, we get that $x=\frac{b}{2y}$. When $y=-1$, we get that $x=-\frac{b}{2}$. When y approaches 0 from below, we get that $\lim_{y\to 0^-}\frac{b}{2y}=-\infty$. Therefore, we get that the layer length of the layer between $y=-1$ and $y=0$ is infinity and that the layer is between $x=-\frac{b}{2}$ and negative infinity.

From expression (3), we can see that the length of a given layer is proportional to the absolute value of b, and that therefore increasing the absolute value of b would increase the length of a "layer".

Because there are infinitely many layers for each graph and the length of each layer increases as $|b|$ increases, then if we increase $|b|$ and fix c, the number of intersecting layers should increase. From this, we postulated that for any given integer r, we can find values for b and c such that $z^2+b\lfloor z\rfloor+c$ has at least r roots, therefore showing that there is no maximum number of roots an equation of this form can have. To prove this, we created a method by which given a number r, we can generate a floored quadratic of the form $z^2+b\lfloor z\rfloor+c$ that has at least r roots. Before we do this, however, we must make some preliminary observations.

Lemma: For the equation $x^2-y^2+b\lfloor x\rfloor+\frac{b^2}{4}=0$, when $b>0$, all points left of the center of the graph (where the center is defined as the point $(-\frac{b}{2},0)$) lie within the boundaries created by equations $y=x+\frac{b}{2}$, $y=-x-\frac{b}{2}$, and $\frac{\left(x+\frac{b}{2}\right)^2}{b}-\frac{y^2}{b}=1$, and that the highest and lowest point in a "layer" ends on one of those three graphs or, in the case of the hyperbola equation, gets infinitely close.

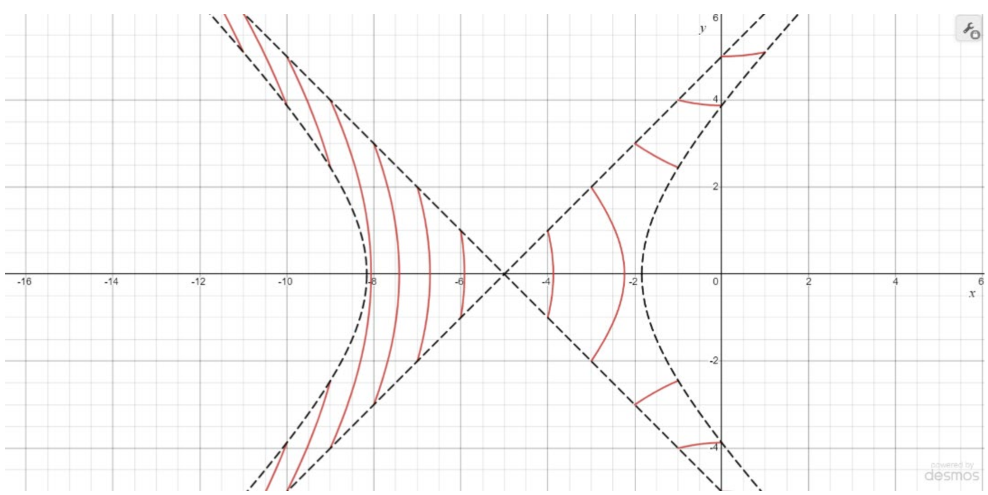


Figure 3. Example plot with bound equations ($\frac{\left(x+\frac{b}{2}\right)^2}{b} - \frac{y^2}{b} = 1, y = x + \frac{b}{2}, y = -x - \frac{b}{2}$) in black and the graph of the equation $x^2 - y^2 + b\lfloor x \rfloor + \frac{b^2}{4} = 0$ in red for $b = 10$.

Proof: We know that the graphs of all the equations mentioned in the lemma are symmetrical about the x-axis, so we will take advantage of this. We will break this proof into two parts. The first part is that the lowest and highest points on a layer will always be on one of the lines provided in the lemma, and the second part will be that some layers will also infinitely approach the hyperbola $x^2 - y^2 + b\lfloor x \rfloor + \frac{b^2}{4} = 0$.

For part 1 we will solve for y from equation given in the lemma.

$$x^2 - y^2 + b\lfloor x \rfloor + \frac{b^2}{4} = 0$$

$$y = \pm\sqrt{x^2 + b\lfloor x \rfloor + \frac{b^2}{4}} \qquad (4)$$

To get bounds from equation (4), we observe that $\left|\sqrt{x^2 + b\lfloor x \rfloor + \frac{b^2}{4}}\right| \leq \left|\sqrt{x^2 + bx + \frac{b^2}{4}}\right| = \left|x + \frac{b}{2}\right|$, giving us $y = x + \frac{b}{2}, y = -x - \frac{b}{2}$ as bounds from equation (4). Note that the positive solution for y from equation (4) is the same as from the first line and the negative solution for y from equation (4) is the same as from the second line if we let $\lfloor x \rfloor = x$. This means that at integer values for x the graph of $x^2 - y^2 + b\lfloor x \rfloor + \frac{b^2}{4} = 0$ intersects one of the lines.

For intersections of the hyperbola, we can plug in one of the equations into the other. First, we solve for y^2 from the equation of the hyperbola stated in the lemma.

$$\frac{\left(x + \frac{b}{2}\right)^2}{b} - \frac{y^2}{b} = 1 \rightarrow y^2 = \left(x + \frac{b}{2}\right)^2 - b$$

Afterwards, we plug in this value for y^2 into $x^2 - y^2 + b\lfloor x \rfloor + \frac{b^2}{4} = 0$.

$$x^2 - \left(x + \frac{b}{2}\right)^2 + b + b\lfloor x \rfloor + \frac{b^2}{4} = 0$$

$$x^2 - \left(x^2 + bx + \frac{b^2}{4}\right) + b + b\lfloor x \rfloor + \frac{b^2}{4} = 0$$

$$-bx + b + b\lfloor x \rfloor = 0$$

$$-x + 1 + \lfloor x \rfloor = 0$$

$$x = 1 + \lfloor x \rfloor$$

This shows us that the hyperbola never intersects with the graph given by equation (2), but that as x increases to approach an integer, the graphs will become infinitely close to intersecting.

From here, maintaining the value of c as $\frac{b^2}{4}$, we will just look at the layer of equation (1) that is between $y = 0$ and $y = -1$. Since we are merely finding a minimum, we can take some shortcuts with the computation of the number of roots from the first infinite length layer of equation (1). To find the number of layers, we will find the number of integer values of x between the left bound, the left vertex of the hyperbola bound and the right bound, the x value of the line $y = x + \frac{b}{2}$ when $y = -1$. The left bound (left vertex of the hyperbola) ensures that the layer of equation (2) that we are counting goes through $y = 0$ and therefore is not completely under or over the layer of equation (1) between $y = 0$ and $y = -1$. The right bound ensures that the layer of equation (2) we are counting goes through $y = -1$. Since we know that the layers that we are counting go through both $y = 0$ and $y = -1$, we know for sure that the layers of equation (2) that we are counting intersect with the infinite length layer of equation (1).

To specify the right bound and find the number of such layers, we will first define point A as the point on the line $y = x + \frac{b}{2}$ where $y = -1$. The x value of Point A is $-1 = x + \frac{b}{2} \rightarrow x = -1 - \frac{b}{2}$. We want to find the number of integer values of x that are between the point labeled A and the left vertex of the hyperbola bound. The x value for the left vertex of the hyperbola is $\frac{\left(x+\frac{b}{2}\right)^2}{b} - \frac{(0)^2}{b} = 1 \rightarrow \left(x+\frac{b}{2}\right)^2 = b \rightarrow x = -\frac{b}{2} - \sqrt{b}$. To find the number of integer values x between these, we take the difference and take the floor. We get a value of $\left\lfloor\left(-1-\frac{b}{2}\right) - \left(-\frac{b}{2} - \sqrt{b}\right)\right\rfloor = \lfloor\sqrt{b}+1\rfloor$.

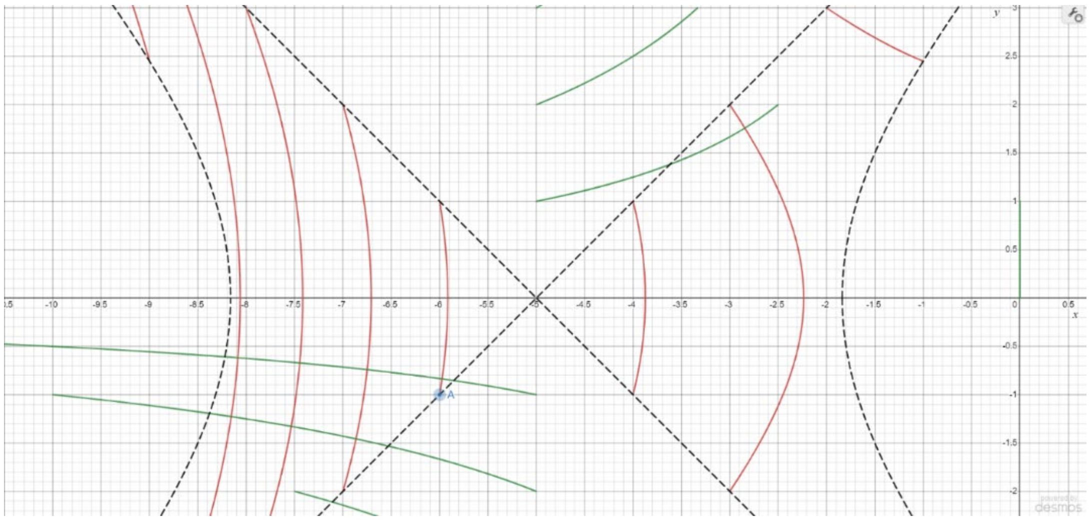


Figure 4. Equations (1) and (2) with bounds with $b = 10$, as well as Point A labeled to help count intersections.

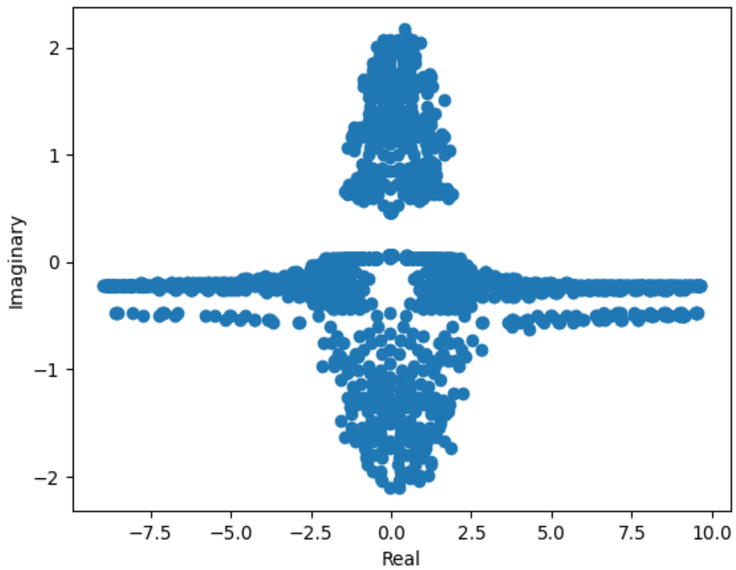
Therefore, for any given b, the floored quadratic $z^2 + b\lfloor z\rfloor + \frac{b^2}{4}$ will have at least $\lfloor\sqrt{b}+1\rfloor$ roots, and therefore there is no upper bound on the number of complex roots for a floored quadratic of the form $z^2 + b\lfloor z\rfloor + c$.

Domain Coloring

As the algebraic method proved faulty, we defaulted upon a computational method in calculating the roots. A common strategy in finding roots of a function is to graph the function. In two dimensions, finding the roots of a function is typically as easy as finding the points at which it intersects the x-axis, and it is this ease that makes graphing an appealing method through which to find roots. However, conventional methods of graphing functions fail when working with complex functions. Complex functions have a two-dimensional input, consisting of a real and imaginary part, and a two-dimensional output, again consisting of a real and imaginary part. Graphing a complex function in the same way one would graph a real function would require creating a four-dimensional graph. To circumvent this problem, we utilized a method commonly used for visualizing complex functions called domain coloring.

In domain coloring, the problem of having too many dimensions is evaded through color. Colors have two dimensions, hue and shade. In domain coloring, the argument of a complex number is represented through their hue, and the modulus of a complex number is represented through its shade[3]. Typically, red is used to represent an argument of 0, and it shifts through the colors of the rainbow (red through violet) as it goes to 2π. For the shade, smaller moduli are darker, with the origin of the complex plane being pure black. For our analysis, we used the python package DColor[4], which ignores the argument of the complex number and only looks at the magnitude. Since we are searching for roots (i.e. when the magnitude is 0), this is acceptable. When a function is plotted using domain coloring, its location on the graph shows the input point, and the color of the point shows the output. This means when looking for roots, we wanted to look for points on the graph that are pure black as it would indicate the input value resulted in the floored quadratic having a value of zero/solution. The input values would then be recorded and graphed, yielding a visual complex (2D to 2D) mapping.

The only part of the graph that is relevant to finding the roots were the dark spots. The domain-colored graph was converted to a black and white image. This black and white graph was then analyzed using computer vision to find the darkest spots in the graph. To make sure roots were not double counted, after a root was found, it and all the surrounding dark pixels were changed to being white, until the ‘finding threshold’ (below which something is not considered a root) was reached. This process was repeated until there were no roots below the threshold in the image of the graph. Because this was done with code, the process for every graph in a certain range was repeated and then graphed. Because domain coloring was not being used for visualization, and only root finding, a method alternative to domain coloring was considered: take the plotted points and just find the smallest one. This comes with its problems: because of how small the values get, floating point error gets compounded. This meant that the error range for finding a root was too high, and this is not a viable strategy (‘rounding error’). With the domain coloring method, there is still some error, but it is human-interpretable and more predictable (‘estimation error’).



Using domain coloring, we created a plot of possible roots for an equation of the form $z^2 + b\lfloor z\rfloor + c$. To accomplish this, we set bounds for the possible values of b and c and started with both b and c on their lower bounds. Then, we graphed the function, found its roots, incremented c by a certain amount, and repeated this process until the upper bound for c, upon which we reset c to its lower bound and incremented b. Continuing this process until both b and c have reached their upper bound, we acquired the following graph. Note that in the figure, the empty space between 0 and approximately 0.5 is created due to the increment of 1. We found roots with imaginary parts in that range, but only ones where in the quadratic either b, c, or both were not integers.

Figure 5. Plot of the possible roots of an equation of the form $z^2 + b\lfloor z \rfloor + c$ with bounds for b and c of -20 to 20 and increment of 1.

Trends in Roots

One way to interpret Figure 5 as presented in the previous section is to see it as $\frac{n}{x}$ and $-\frac{n}{x}$ graphs. Notice earlier that only equation (2) has c in it, meaning that only one of them changes according to the value of c. As seen below in Figure 6, when we increase the value of c, we find that the graph of equation (2) shifts rightward. Equation (2) takes the shape of an either vertical or horizontal hyperbola due to the equation being roughly of the form If we were to increase the value of c sufficiently, the graph of equation (2) would turn into a vertical hyperbola (vertices of the hyperbola vertical to each other).

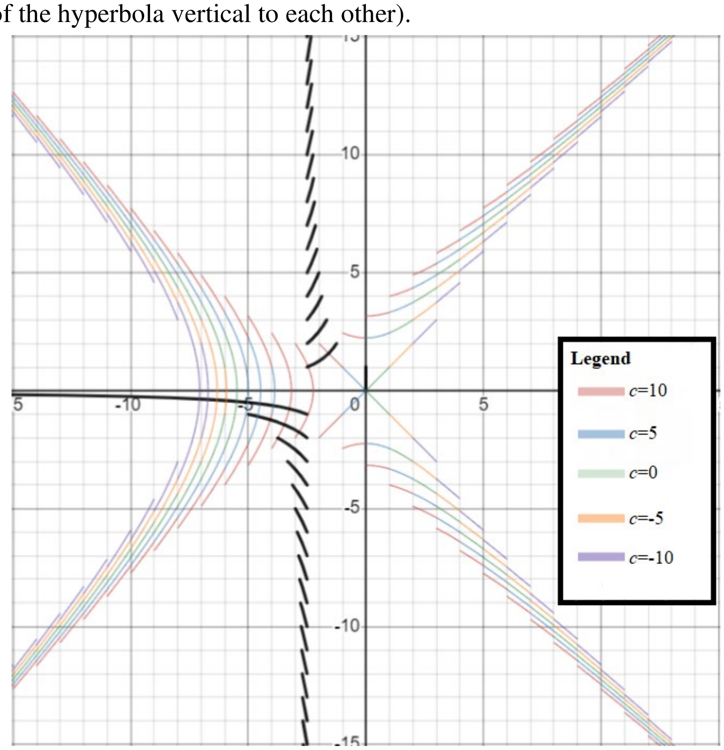


Figure 6. Changes in the plot of equation (2) as the value of c changes.

What this tells us is that when we hold the value of b constant while changing the value of c, the intersections of the two graphs trace out most of the shape of the many fragments of $\frac{n}{x}$ graphs that is the graph of equation (1). This explains why in Figure 3 the plot seems to be made up of many $\frac{n}{x}$ functions plotted together as the different values of c account for the different n's in the $\frac{n}{x}$'s.

The behavior of the roots when incrementing b, however, is difficult to define with precision. When increasing b, both the plots of equations (1) and (2) generally shift to the left, like as seen in the graph of a regular quadratic, although the effect this has on the roots is unpredictable due to the ways the “layers” of the graphs of equation (1) and (2) shift.

Conclusion

In this paper, we explored finding complex roots of a quadratic equation with floor functions in it, along with the behaviors of the roots as coefficients within the quadratic were changed. We first explored using algebra to simplify the plot of the floored quadratic, taking the root as the intersection of two separate 2-dimensional graphs. Using this method, we were able to prove that there is no upper bound for the number of complex roots that an equation of the form $z^2 + b\lfloor z \rfloor + c$ can have. Additionally, we also utilized domain coloring of the graph as a method to quickly approximate the roots given many equations of the form $z^2 + b\lfloor z \rfloor + c$. Using this graph and analysis of the behavior, we found that as the value of c changed, the roots traced out the rough shape of a hyperbola seen when graphing a $\frac{n}{x}$ function. Changing b generally shifted the roots to the right, although its effect on the roots of a $z^2 + b\lfloor z \rfloor + c$ function was much more unpredictable and difficult to describe with precision.

Limitations and Further Research

There was little to no past research on the topic that we could find, and as such, space for further research is plentiful. There are multiple avenues of further research available, both in terms of refining the approaches for analyzing these floored quadratics we presented in this paper and also in broadening the scope of study when it comes to floored quadratics.

Finding roots using domain coloring produces estimation error for most roots. Future studies could possibly utilize the “root” produced by domain coloring as a starting point, and then taking advantage of how floored quadratics are analytic on each individual unit square by trying to use Newton’s Method to get a more precise root while ensuring that the point stays within the unit square of the initial estimation. Such a method was too computationally heavy to be utilized by us but may yield much more accurate data that could expand the analysis and understandings of the graphs produced in this paper. We also believe that with a more accurate algorithm, one could also begin to analyze the correlation between b, c, and the number of roots a floored quadratic has. We initially wanted to create a 3D graph with the x and y axes denoting b and c and the z axis denoting the number of roots for the floored quadratic $z^2 + b\lfloor z \rfloor + c$, but we decided that there was too much error in the domain coloring analysis to be able to create such a graph.

In addition to the limitations of domain coloring, there are also multiple avenues for further research in terms of what kind of quadratics we deal with. In the beginning of this paper, we defined the floor of a complex number as that number with its real and imaginary components rounded down to the nearest integer. However, there are multiple possible definitions for the floor of a complex number. One such possibility would be that the floor of a complex number is defined as having the same argument but having the modulus rounded down to the nearest integer. We speculate that domain coloring would still work in this case, but we have not tried it. In addition, there are also many ways that the floor function could be implemented into a quadratic that we did not explore here, such as $\lfloor z^2 + bz \rfloor + c = 0$ or other similar expressions. We also did not deal with complex coefficients for b and c, as although they were possible to root plot through domain coloring, we were unable to meaningfully interpret any of the plots generated, as they seemed to be randomly scattered across the complex plane.

Acknowledgments

We would like to thank the staff of the math department of the Governor's Honors Program. The initial research for this paper was conducted during the program with the assistance of the staff of the math department, and without the Governor's Honors Program or the aid of the program staff this paper would not have been possible. We would also like to give a special acknowledgement to our main research advisor, Bill Shillito, who provided invaluable assistance during the Governor's Honors Program for this research project, most notably through suggesting the use of domain coloring.

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