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Deriving Equations of Motion Using Euler-Lagrange Equation and Determining Efficiency Factors of a Trebuchet

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ABSTRACT

A Trebuchet is a non-holonomic system which is modelled after a catapult. Lagrangian mechanics was a formulation introduced by Professor Joseph Louis Lagranage which predicts the minimum path a body follows during its motion of travel. In this paper, I built a wooden trebuchet to experiment with its shooting range under different counterweight loads. The trebuchet model was customized by adding a sling to its swing arm which changes the dynamics of the setup as the sling essentially adds scope to the trebuchet's arm increasing its rotational energy. The paper compares readings based on several counterweight settings and uses and uses the Euler-Lagrange equation to analyze the shooting ranges of the trebuchet. By considering the several frictional forces, we will derive the three equations of motion which will solely depend on the degrees of freedom of the system and will determine how efficient the Trebuchet is for a particular counterweight.

Introduction

A Brief Historical Background of Trebuchet

The trebuchet is a medieval siege warfare machine that uses a counterweight and a long arm to propel projectiles. According to history, it was invented in China around 300BC, which used labor to sway the arm. Later it was used by European armies to either defend against enemies or destroy walls and forts belonging to their enemies. There was also an amelioration to the trebuchet with a hinged counterweight to release heavier projectiles at a certain angle to help increase the range and velocity of the projectile. However, ever since the inception of gunpowder in the 9th century, the trebuchet was a bygone weapon and soon cannons replaced them as they proved to be more efficient.

Modifications to the Trebuchet

A basic trebuchet can be modified structurally to experiment with its efficiency factors. The addition of a sling to the trebuchet changes its dynamics as the sling essentially adds scope to the trebuchet's arm increasing its rotational energy: $E_R = \frac{1}{2}\left(\sum_j m_j r_j^2\right)\omega^2$.

The sling creates a greater angular velocity while launching the projectile:

$$\omega = \frac{d\theta}{dt} = \frac{l_2 m_b g\cos(180-\phi) + l_1 T_2 \cos(\phi) - l_2 T_1 \cos(\psi)}{I_b}$$

where m_b is the mass of the beam, I_b is the moment of inertia of the beam, T_1 is the tension of the sling, and T_2 is the tension of the counterweight rope.

Changing the counterweight mass could copiously influence the range of a trebuchet. In this paper, we shall compare the readings of several counterweight settings and use Lagrangian mechanics to show how a change in the mass of a counterweight changes the range of the trebuchet. We will use Euler- Lagrange equation, a way to solve the equations of motion, with limited degrees of freedom (θ, ψ, and ϕ). This paper will illustrate the findings with a realistic model of a trebuchet and a virtual model. The general Euler-Lagrange equation, as a function of the generalized coordinates, is:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

Furthermore, we will define three equations of motion and prove that the Lagrangian function (T - V or Kinetic Energy - Potential Energy) will be 0. The potential energy of the counterweight transfers to the projectile as kinetic energy as the latter is launched into air in its trajectory.

Geometry and Equations

Here is a two-dimensional representation of the trebuchet with important geometric parameters labelled accordingly. These parameters will be used in derivations of equations of motion and mathematical expressions.

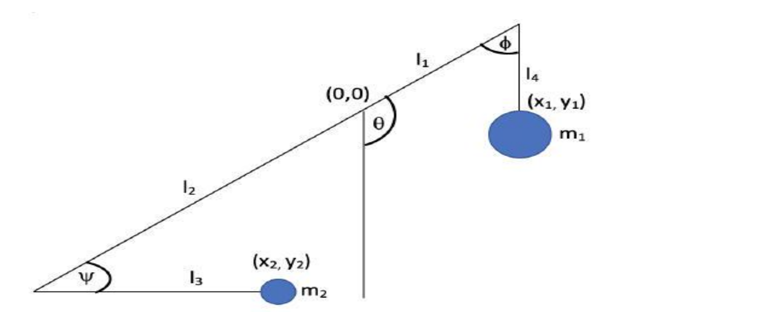

Figure 1. A two-dimensional representation of the trebuchet.

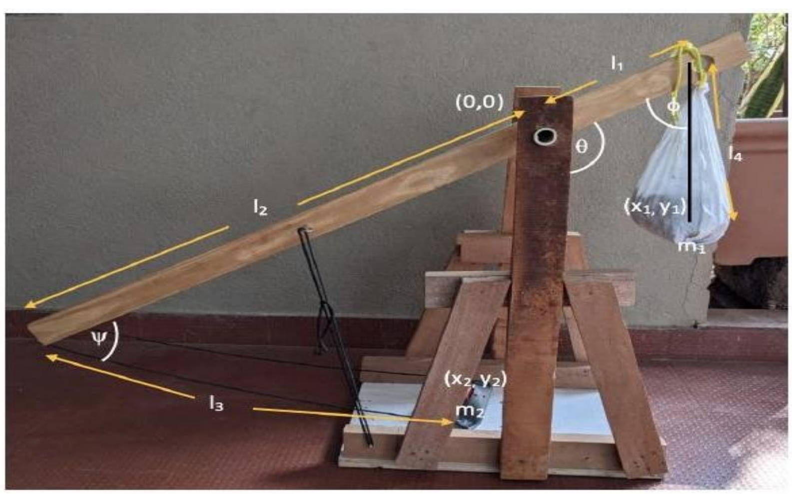

Figure 2. A picture of the trebuchet built for the project.

m1: Mass of Counterweight
m2: Mass of projectile

l1: Length of Short Arm
l2: Length of Long Arm
l3: Length of Projectile Sling
l4: Length of Counterweight Sling
θ: Angle subtended by Short Arm and Frame
ϕ: Angle subtended by Short Arm and Counterweight Sling
ψ: Angle subtended by Long Arm and Projectile Sling
(x1, y1): Position of Counterweight
(x2, y2): Position of Projectile
The positions of the masses (*m1* and *m2*) in the trebuchet are:
$\vec{R}_1 =< l1\sin(\theta) - l4\sin(\theta+\varphi), -l1\cos(\theta) + l4\cos(\theta+\varphi) >$
$\vec{R}_2 =< -l2\sin(\theta) - l3\sin(-\theta+\psi), l2\cos(\theta) - l3\cos(-\theta+\psi) >$

Differentiating the masses, the velocities of the trebuchet are:

$$\vec{R}_1 =< l_1\dot{\theta}\cos(\theta) - l_4\left(\left(\dot{\theta}\cos(\theta)\cos(\phi) - \dot{\phi}\sin(\theta)\sin(\phi)\right) + \left(-\dot{\theta}\sin(\theta)\sin(\phi) + \dot{\phi}\cos(\theta)\cos(\phi)\right)\right)$$
$$, l_1\sin(\theta) + l_4\left(\left(-\dot{\theta}\sin(\theta)\cos(\phi) - \dot{\phi}\cos(\theta)\sin(\phi)\right) - \left(\dot{\theta}\cos(\theta)\sin(\phi) + \dot{\phi}\sin(\theta)\cos(\phi)\right)\right) >$$
$$\vec{R}_2 =< -l_2\dot{\theta}\cos(\theta) - l_3\left(\left(\dot{\psi}\cos(\psi)\cos(\theta) - \dot{\theta}\sin(\psi)\sin(\theta)\right) - \left(-\dot{\psi}\sin(\psi)\sin(\theta) + \dot{\theta}\cos(\psi)\cos(\theta)\right)\right)$$
$$, -l_2\dot{\theta}\cos(\theta) - l_3\left(\left(-\dot{\psi}\sin(\psi)\cos(\theta) - \dot{\theta}\cos(\psi)\sin(\theta)\right) + \left(\dot{\psi}\cos(\psi)\sin(\theta) + \dot{\theta}\sin(\psi)\cos(\theta)\right) >\right)$$

The Lagrangian function in terms of Cartesian Coordinates:

$$L = m_1(\dot{x}_1^2 + \dot{y}_1^2)/2 + m_2(\dot{x}_2^2 + \dot{y}_2^2)/2 + m_b\dot{\theta}^2(l_1^2 - l_1l_2 + l_2^2) - m_1y_1(\theta,\phi)g - m_2y_2(\theta,\psi)g + (l_1m_b\cos(\theta)g)/2 - (l_2m_b\cos(\theta)g)/2$$

The Lagrangian function with negligible arm mass would be:

$$L = T - V = \left(l_1^2m_1\dot{\theta}^2\right)/2 - l_1l_4m_1\dot{\theta}^2\cos(\phi) + l_1m_1g\cos(\theta) - l_1l_4m_1\dot{\theta}\dot{\phi}\cos(\phi) + l_4^2m_1\dot{\theta}\dot{\phi} + \left(l_4^2m_1\dot{\theta}^2\right)/2 + \left(l_4^2m_1\dot{\phi}^2\right)/2 - l_4m_1g\cos(\theta+\phi) + (l_2^2m_2\theta^2)/2 - l_2m_2g\cos(\theta) - l_2l_3m_2\dot{\theta}^2\cos(\psi) + l_2l_3m_2\theta\dot{\psi}\cos(\psi) + \left(l_3^2m_2\dot{\psi}^2\right)/2 - l_3^2m_2\dot{\theta}\psi + \left(l_3^2m_2\dot{\theta}^2\right)/2 + l_3m_2g\cos(\psi - \theta)$$

When we solve the Euler-Lagrange equation with respect to θ:

Equation 1: $\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$

The first expression of motion with negligible mass is:

Expression 1: $m_1\left(l_1g\sin(\theta) + l_4g\sin(\theta)\cos(\phi) + l_4g\sin(\phi)\cos(\theta) - l_1^2\ddot{\theta} + 2l_1l_4\ddot{\theta}\cos(\phi) - 2l_1l_4\dot{\theta}\dot{\phi}\sin(\phi) + l_1l_4\ddot{\phi}\cos(\phi) + l_1l_4\dot{\phi}^2\sin(\phi) + 2l_1l_3\ddot{\theta}\cos(\psi) - l_4^2\ddot{\phi} - l_4^2\ddot{\theta}\right) + m_2(l_2g\sin(\theta) - l_3g\cos(\psi)\sin(\theta) + l_3g\cos(\theta)\sin(\psi) - l_2^2\ddot{\theta} - 2l_2l_3\dot{\theta}\dot{\psi}\sin(\psi) - l_2l_3\ddot{\psi}\cos(\psi) + l_2l_3\dot{\psi}^2\sin(\psi) + l_3^2\ddot{\psi} - l_3^2\theta\right)$

When we solve the Euler-Lagrange equation with respect to ϕ, we get:

Equation 2: $\frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0$

Accordingly, the second expression of motion with negligible mass is:

Expression 2: $m_2\left(l_2l_3\dot{\theta}^2\sin(\psi) - l_3g\cos(\theta)\sin(\psi) - l_2l_3\ddot{\theta}\cos(\psi) + l_3g\sin(\theta)\cos(\psi) + l_3^2\ddot{\theta} - l_3^2\ddot{\psi}\right)$

When we solve the Euler-Lagrange equation with respect to ψ, we get:

Equation 3: $\frac{\partial L}{\partial \psi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) = 0$

Therefore, the third expression of motion with negligible mass is:

Expression 3: $m_1\left(l_1 l_4 \dot{\theta}^2 \sin(\phi) + l_4 g \cos(\theta) \sin(\phi) + l_4 g \cos(\phi) \sin(\theta) + l_1 l_4 \ddot{\theta} \cos(\phi) - l_4^2 \ddot{\theta} - l_4^2 \ddot{\phi}\right)$

Results from the Experiments

The trebuchet is a is a non-holonomic constraint system with dependent, independent, and control variables.

Variables

Independent variable: Counterweight (m_2)
Dependent variable: Range of projectile
Control variables: listed in Table 1 below

Table 1. Control variables in the experiment.

Dimensions of parts of the Trebuchet	
l_1 (length of short arm)	0.28 m
l_2 (length of long arm)	0.91 m
l_3 (length of sling)	0.79 m
l_4 (length of counterweight sling)	0.14 m
m_2 (mass of projectile)	0.052 kg

During the experiment it was arduous to control the angles subtended in the Trebuchet so we will assume that θ, ϕ, and ψ are constant angles:

$$\psi = 48°, \theta = 119°, \phi = 55°$$

Results

Table 2. Measurements of the distance traveled by the projectile over 5 trials under 8 different counterweight conditions.

Counterweight in kgs	Range of Projectile (Distance) in meters				
	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5
2	2.92	2.21	3.10	3.35	2.64
3	7.37	7.01	6.91	7.01	6.98
4	9.55	10.11	8.83	9.50	9.11
5	11.48	12.40	11.00	11.38	11.72
6	12.67	13.46	12.65	12.75	13.36
7	14.87	15.05	15.12	16.46	15.85
8	18.34	17.80	18.21	18.49	17.74

9	18.92	19.54	19.95	20.01	19.31

Table 3. The average range and standard deviation of the projectile modelled by the trebuchet.

Range of projectile		
Counterweight in kgs	**Average Distance in meters**	**Standard Deviation**
2	2.84 ± 0.57	0.44
3	7.06 ± 0.23	0.18
4	9.42 ± 0.64	0.48
5	11.60 ± 0.70	0.52
6	12.98 ± 0.40	0.40
7	15.47 ± 0.80	0.67
8	18.11 ± 0.38	0.33
9	19.55 ± 0.54	0.45

A Range versus Counterweight graph

Here is a graph representing the values measured in the trials during the experiment.

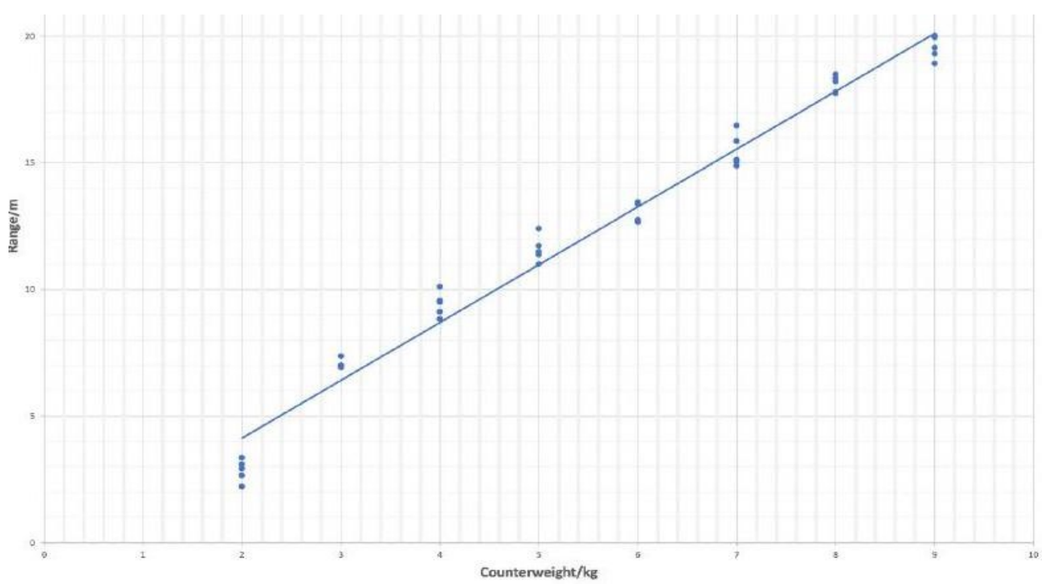

Figure 3. Graph of Range in meters on Y-axis to Counterweight in kgs on X-axis.

Discussions

From figure 3, we learn that the projectile range measurements had the least variation when the counterweight was 6kg. Inputting the values when the counterweight was 6kg and using the dimensions of the trebuchet in the three equations of motion, we get:
Equation 1 $= 16.08$
Equation 2 $= 0.3839$
Equation 3 $= 0.9564$

As we can see from the data, the values of the equation are not equal to 0 but are relatively close to the desired value. This can be because the Lagrangian function did not consider the frictional forces and haply the strength, suspension, material of the string. The release angle of the trebuchet also varied between launches and was not constant as it was onerous to maintain it at an optimal degree.

Efficiency of Trebuchet			
Counterweight in kgs	Average Measured Value in meters	Theoretical Value in meters	Efficiency
2	2.84	15.86	17.91%
3	7.06	23.81	29.65%
4	9.42	29.66	31.76%
5	11.60	33.23	34.91%
6	12.98	36.01	36.04%
7	15.47	40.80	37.92%
8	18.11	44.93	40.31%
9	19.55	49.86	39.21%

Conclusion

The efficiency, $\frac{\text{Average-Measured-Value}}{\text{Theoretical-Value}} 100\%$ and the theoretical values incorporated frictional forces (gravity: $9.81\ \mathrm{m/s^2}$ and wind: $2.1\ \mathrm{m/s^2}$) and *inertia* of the throwing arm and *inertia* of the counterweight. However, the Lagrangian function and the three equations of motion did not include the *inertia* of the system.

Limitations

Even though the results show that the trebuchet was most efficient when the counterweight was 8 kg, the system was unstable in its position with higher loads (weights). While performing experiments for counterweights larger than 8 kilograms, the trebuchet appeared to sway and tilt about its position while launching the projectile. I realized that the design, the dimensions of the model, and the material used to construct the trebuchet were appropriate for counterweights in the range of 2 to 8 kg.

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