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Partial Solution-Preserving Integrable Generalization Method for Autonomous ODE Systems

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ABSTRACT

In this paper, we propose a method for the generalization of some generic fundamental, abstract differential equations into generalized systems. We hypothesize that these generalized systems are fit to model some real-life phenomena, which can be of practical interest. We confirm our hypothesis by considering examples that are known to be confirmed with the experiment as well as examples that are still to be discovered.

Introduction

Several scientific principles and more generally real-life situations are concerned with the relationships between varying quantities. Because derivatives are used to illustrate rates of change in mathematics, such principles are frequently characterized in terms of linear and non-linear differential equations (Ming). In order to describe real-life situations by differential equations we must first identify the problem that needs to be solved. Then, by applying a number of assumptions we convert the real-life problem into a set of differential equations (Trench). Finally, we should be able to solve them analytically, or even numerically, although the special attention is often paid specifically to integrable cases (Newell).

Therefore, alongside the goal of finding the *solutions* to known mathematical models, we could also consider another objective (in a certain sense an inverse to the previous one) of finding the *equations* that are known to have a specific kind of solutions. This is the essence of our approach, which we suggest in this paper.

From a practical point of view, the latter may even be more interesting. Let's give an example of such a problem: suppose we have a model equation describing free oscillations or the propagation of an impulse of a desired spectral quality in a medium without attenuation or distortion of its shape. As a rule, such an equation (which we will call “generic”) is a fairly simple autonomous integrable ordinary differential equations (ODE). From a practical point of view, for example, to create electrical or sound generators of corresponding oscillations or impulses it is necessary to have some form of “generalized” model, which includes the effects of dissipation and driving process. Specific examples of the application of our proposed approach are discussed in the following sections. Other interpretations of the “generalized” model are possible as well, for example, as synchronized state in coupled oscillator systems (Joshi, Sen and Kar). Similar problems have also been considered in the feedback theory (Boulite, Hadd and and Maniar). However, although the approaches used there are universal, they are also complex as well. Often, it is impossible to obtain the exact solutions using the feedback theory, so it is necessary to apply other methods such as the perturbation theory (Bender and Orszag). Although our proposed approach is perhaps less universal, it doesn’t require the approach of perturbation theory, and it is possible to obtain exact solutions with it (meaning integrable).

Thus, our proposed approach can be mathematically formulated as solution preserving generalization of integrable autonomous ODE. We hypothesize that sets of differential equations built with such an approach are fit to model some real-life phenomena (including undiscovered yet), the examples to which are given in the following sections.

Mathematical Model

In this section, our approach is considered for the third order of autonomous ODE. However, such approach is valid for arbitrary order of autonomous ODE as well and all the statements made in this section are relevant to autonomous ODE of arbitrary order. Autonomous system of ODE of third order can be represented as:

$$\begin{cases}\dot{x}_1 = f_1(x_1, x_2, x_3)\\ \dot{x}_2 = f_2(x_1, x_2, x_3)\\ \dot{x}_3 = f_3(x_1, x_2, x_3)\end{cases} \qquad (1)$$

where $\{f_1, f_2, f_3\}$ are known functions of x_1, x_2, x_3; $x_i = x_i(t)$ $(i = 1,2,3)$ are sought-for functions of t. $\dot{x}_i$ represents the derivative with respect to time. The system (1) is autonomous since its right-hand side does not depend on time explicitly. The solutions of the system (1) are the set of functions $x_1(t), x_2(t), x_3(t)$, which upon substituting turn each equation in the system (1) to the true equality.

A scalar function $U(x_1, x_2, x_3)$ is known to be integral of the system (1) if it is not an identical constant, but becomes a constant at each solution of system (1). The equality:

$$U(x_1, x_2, x_3) = c, \quad (\text{c} = \text{const}) \qquad (2)$$

is known as the first integral of the system (1).

Along with system (1), we consider an autonomous system of third-order ODE:

$$\begin{cases}\dot{x}_1 = f_1(x_1, x_2, x_3) + g_1(x_1, x_2, x_3)\\ \dot{x}_2 = f_2(x_1, x_2, x_3) + g_2(x_1, x_2, x_3)\\ \dot{x}_3 = f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)\end{cases} \qquad (3)$$

where $\{g_1, g_2, g_3\}$ are known functions of x_1, x_2, x_3;

Let the set $\{x_1^0,\ x_2^0,\ x_3^0\}$ be a partial solution to the system (1), and

$$U(x_1^0,\ x_2^0,\ x_3^0) = \text{c} \qquad (4)$$

Then a necessary and sufficient condition that the set of functions $\{x_1^0,\ x_2^0,\ x_3^0\}$ will be simultaneously the solution of system (1) and (2) is be the ability to represent system (3) in the form:

$$\begin{cases}\dot{x}_1 = f_1(x_1, x_2, x_3) + F_1(U(x_1, x_2, x_3) - c)\\ \dot{x}_2 = f_2(x_1, x_2, x_3) + F_2(U(x_1, x_2, x_3) - c)\\ \dot{x}_3 = f_3(x_1, x_2, x_3) + F_3(U(x_1, x_2, x_3) - c)\end{cases} \qquad (5)$$

where $\{F_1, F_2, F_3\}$ are well-behaved functions, which meet the condition:

$$F_i(0) = 0, \text{ where } i = 1, 2, 3 \qquad (6)$$

Let us now show it. Substituting $\{x_1^0,\ x_2^0,\ x_3^0\}$ into the system (5) and using (4) results in:

$$\begin{cases}\dot{x}_1^0 = f_1(x_1^0, x_2^0, x_3^0) + F_1(U(x_1^0, x_2^0, x_3^0) - c)\\ \dot{x}_2^0 = f_2(x_1^0, x_2^0, x_3^0) + F_2(U(x_1^0, x_2^0, x_3^0) - c)\\ \dot{x}_3^0 = f_3(x_1^0, x_2^0, x_3^0) + F_3(U(x_1^0, x_2^0, x_3^0) - c)\end{cases}$$

Since the set $\{x_1^0,\ x_2^0,\ x_3^0\}$ is a partial solution of the system (1), we get:

$$\begin{cases}F_1(U(x_1^0, x_2^0, x_3^0) - c) = 0\\ F_2(U(x_1^0, x_2^0, x_3^0) - c) = 0\\ F_3(U(x_1^0, x_2^0, x_3^0) - c) = 0\end{cases}$$

Or

$$\begin{cases} F_1(0) = 0 \\ F_2(0) = 0 \\ F_3(0) = 0 \end{cases}$$

by assumption (6) the last equality is true.

Let's apply this approach to a well-known linear undamped oscillator ODE (Zill).

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1, \end{cases} \tag{7}$$

Then, one of the corresponding generalized system can be represented in the form:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 - x_2 \cdot \left(\frac{x_1^2}{2} + \frac{x_2^2}{2} - 1\right) \end{cases} \tag{8}$$

Where $\frac{x_1^2}{2} + \frac{x_2^2}{2}$ is the first integral of the system (integral of motion of linear undamped oscillator), 1 – some specific value of this integral, the term $-x_2 \cdot \left(\frac{x_1^2}{2} + \frac{x_2^2}{2} - 1\right)$ in the second equation of the system is a function F, satisfying the condition that $F(0) = 0$. As it can be easily seen, the added term $\left(\frac{x_1^2}{2} + \frac{x_2^2}{2} - 1\right)$ in the second equation of the system vanishes on the phase trajectory corresponding to the integral of motion equaling to 1. Thus, generic system (7) and generalized system (8) have a common solution corresponding to the integral of motion equaling to 1. This is illustrated with phase diagram representing phase spaces for generic and generalized systems.

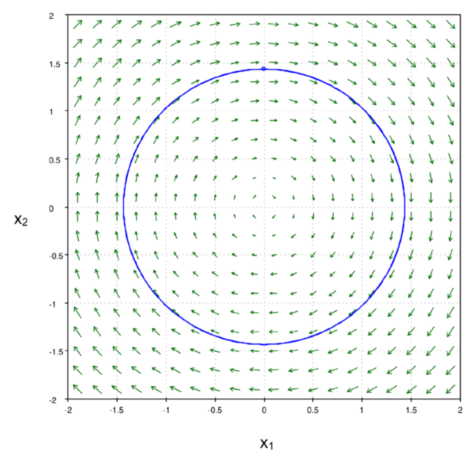

Figure 1. Phase trajectory of generic system (7) corresponding to the value of the first integral equaling to 1.

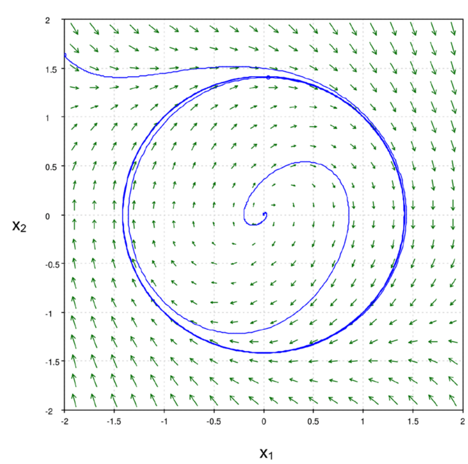

Figure 2. Phase trajectory corresponding to the stable limit cycle is phase space of the generalized system (8).

As can be seen, phase trajectories for these systems are generally different excluding the trajectory corresponding to the value of integral of motion equaling to 1. The closed trajectory of the generalized system corresponds to the limit cycle in the phase space. For the chosen specific function F, this limit cycle is stable.
System (8) can be considered as a specific case to the known Rayleigh-van der Pol harmonic oscillator equation system , introduced in (AMOS). In the work (Buldakov, Samochetova and Sitnikov), such type of equations is used as a mathematical model for the organized behavior in the cardiovascular system.

The right hand side of the second equation of the system (8) includes a generalizing term $-x_2 \cdot \left(\frac{x_1^2}{2}+\frac{x_2^2}{2}-1\right)$, which can be interpreted as a combination of terms representing "damping" and "driving" forces. Moreover, "damping" force is non-linear and depends on the coordinate as well as velocity. This leads to the question: is it possible for a generalization procedure, in which the generalization term (in particular "damping" force) has a simpler and conventional form?
For that, let's consider the special case of an autonomous ODE of third-order:

$$\begin{cases} \dot{x}_1 = f_1(x_1,x_2,x_3) + g_1(x_1,x_2,x_3), \\ \dot{x}_2 = f_2(x_1,x_2,x_3), \\ \dot{x}_3 = f_3(x_1,x_2,x_3), \end{cases} \tag{9}$$

Where, in general, $g_1(x_1,x_2,x_3)$ does not have to be identical to $F_1(U(x_1,x_2,x_3)-c)$, as it was necessary for (5).
Let's find the condition for the system (1) and (9) to have a common solution for the dependent variable $x_1(t)$. For that, suppose that by substituting variables in system (9) as:

$$\begin{matrix} x_1 \to x_1 \\ x_2 \to X_2, \\ x_3 \to X_3, \end{matrix} \tag{10}$$

we are able to transform (9) to the form:

$$\begin{cases} \dot{x}_1 = F_1(x_1,X_2,X_3), \\ \dot{X}_2 = F_2(x_1,X_2,X_3), \\ \dot{X}_3 = F_3(x_1,X_2,X_3) + G_1(U(x_1,X_2,X_3)-c), \end{cases} \tag{11}$$

Then, if $G_1(0)=0$, systems (1) and (9) would have a common solution for the dependent variable $x_1(t)$.

Results and Applications

Application 1: Dumped and Driven Self-Induced Transparency Equations

The effects of highly energetic ultra-fast pulses propagating through a resonant medium are not adequately explained by the linear dispersion theory. In such extreme situation, the interaction between the propagating pulses and the resonant medium is very efficient that the system evolves into a self-consistent state. Then after, light propagates into the resonant medium with no major modification in their shapes. This is the self-induced transparency (SIT) regime, also known as the McCall-Hahn soliton regime (McCall and Hahn). The physics of SIT can be understood by taking into account the dynamics of a pulse's interaction with the medium. If the pulse energy is high all the particles of the medium are found in the upper state. Thus, the medium is completely inverted. The remaining pulse leads the fully inverted particles to release stimulated light, returning energy to the propagating field. The excited particles are rapidly relaxed to the lower state in this manner. Under these conditions, a pulse can propagate through an absorbing media with no losses, constantly wasting and recovering its energy. The standard approach to describe the interaction between ultra-fast pulses with a two levels system is based on the slowly varying envelope approximation (SVEA) (Allen and Eberly).

Bullough et al. (Bullough, Caudrey and Eilbeck) derived a set of equations describing resonant interaction of pulses with a two-levels medium. These SIT equations that describe normalized real magnitudes of electric field E, polarization P and population inversion N with for short pulses are

$$K\,\dot{E} = P \qquad (12)$$

$$\dot{P} = E\,N \qquad (13)$$

$$\dot{N} = -E\,P \qquad (14)$$

Where the dot denotes differentiation with respect to time $(t - \frac{z}{u})$ and u has the dimension of velocity. The pulse is considered to propagate in the positive z direction with a phase velocity u. K is defined as $\frac{c-1}{u}$ where c is the speed of light in vacuum.

From equations (12) and (14):

$$\dot{N} = -E\,K\,\dot{E} = -\frac{K}{2}(\dot{E^2}) \qquad (15)$$

The integral of equation (15) gives:

$$N = -\frac{K}{2}E^2 + C_1 \qquad (16)$$

Since initially, at time $t = 0$, the boundary conditions for a partial case of single soliton solutions are:

$$E(0) = 0$$
$$N(0) = -1 \qquad (17)$$

By replacing the initial conditions in equation (16) we obtain the value of the integration constant C_1:

$$N(0) = -\frac{K}{2}E^2(0) + C_1 \qquad (18)$$

Hence the integration constant C_1 is equal to -1.

Substituting equation (16) into (13) gives:

$$\dot{P} = -\frac{K}{2}E^3 - E \qquad (19)$$

Differentiating equation with respect to time (12) gives:

$$K \cdot \ddot{E} = \dot{P} \qquad (20)$$

Substituting (19) into (20), a first integral of SIT equations system Eq.12 – Eq.14 can be found in the form of undamped and unforced Duffing equation (Bender and Orszag).

$$\ddot{E} + \frac{1}{K} \cdot E + \frac{1}{2} \cdot E^3 = 0 \qquad (21)$$

Equation (21) has a soliton-like solution describing pulses with area under envelope of E equal to 2π (so-called $2\pi -$ pulse). Pulses whose areas are multiples of 2π propagate in a two-level medium with no changes in their envelopes (the soliton propagation regime). The solution of the electric field is:

$$E = \frac{2}{\sqrt{-K}} \cdot sech\left(\frac{t}{\sqrt{-K}}\right) \qquad (22)$$

Now let us consider a perturbed SIT equations system. To obtain that, one should also add to (12) – (14) a term P' representing a loss and a driving force:

$$K\,\dot{E} = P + P' \qquad (23)$$

$$\dot{P} = E\,N \qquad (24)$$

$$\dot{N} = -E\,P \qquad (25)$$

Introducing new variable with tilde:

$$\tilde{P} = P + P' \qquad (26)$$

$$P = \tilde{P} - P' \qquad (27)$$

Substituting equation (27) into (24) we obtain:

$$\dot{\tilde{P}} - \dot{P}' = E\,N \qquad (28)$$

Substituting equation (27) into (25) we obtain:

$$\dot{N} = -E\,\tilde{P} + E\,P' \qquad (29)$$

From equation (23):

$$K\,\dot{E} = \tilde{P} \qquad (30)$$

From equation (28):

$$\dot{\tilde{P}} = E\,N + \dot{P}' \qquad (31)$$

Introducing new variable with tilde and looking for solution at non-zero E:

$$E\,N + \dot{P}' = E \cdot \tilde{N} \qquad (32)$$

$$\tilde{N} = N + \frac{\dot{P}'}{E} \qquad (33)$$

$$\dot{\tilde{N}} = \dot{N} + \frac{d}{dt}\left(\frac{\dot{P}'}{E}\right) \qquad (34)$$

Substituting from (28):

$$\dot{\tilde{N}} = -E \cdot \tilde{P} + \frac{d}{dt}\left(\frac{\dot{P}'}{E}\right) + E \cdot P' \qquad (35)$$

Then the perturbed RMB (23) – (25) can be re-written as:

$$K \cdot \dot{E} = \tilde{P} \qquad (36)$$

$$\dot{\tilde{P}} = E \cdot \tilde{D} \qquad (37)$$

$$\dot{\tilde{D}} = -E \cdot \tilde{P} + \frac{d}{dt}\left(\frac{\dot{P}'}{E}\right) + E \cdot P' \qquad (38)$$

where full derivative with respect to t is denoted as $\frac{d}{dt}$. Obtained equations system (36) – (38) constitutes the (11) condition in a general form.

Let's consider a special case by assuming the P' as a non-linear loss and constant driving force

$$P' = -(A \cdot E) \cdot E + g \qquad (39)$$

where A and g are constants.

Then the (36) – (38) can be re-written as:

$$K\,\dot{E} = \tilde{P} \qquad (40)$$

$$\dot{\tilde{P}} = E \cdot \tilde{N} \qquad (41)$$

$$\dot{\tilde{N}} = -E \cdot \tilde{P} + (2 \cdot A \cdot \ddot{E} + A \cdot E^3 + g \cdot E) \qquad (42)$$

i.e., it is the form of (11), if:

$$K = -\frac{2 \cdot A}{g} \qquad (43)$$

with boundary conditions $E(-\infty) = 0, \dot{E}(-\infty) = 0$ (44)

It can be concluded that the perturbed SIT equations system (23) – (25) with boundary conditions (44) still describes a soliton-like pulse with area under envelope of E equal to 2π ($2\pi -$ pulse), velocity and amplitude and duration of which are dependent on parameters of non-linear loss and a driving force (38); the latter can be realized by an external continuous wave with second harmonic generation at low conversion (Boyd). As far as we know, dumped and driven RMB equations system was not considered previously.

Application 2: Generalization of The Developed Approach to a Dumped and Driven Reduced Sine-Gordon Equations

As it is noted in Bullough et al. (Bullough, Caudrey and Eilbeck), the RMB equations system can be transformed to reduced sine-Gordon (RSG) equations system by replacing P and N in (12)-(14) by:

$$P = sin\,(\varphi)\ and\ \ N = cos(\varphi) \qquad (45)$$

The results obtained earlier for RMB equations system are relevant also to RSG equations system. Analytical solutions with a partial approach, when the loss and the driven force terms are pre-assumed from start in the form of equation (39) are reported in (Costabile and Parmentier) and (Pedersen and Saermark) in the case of Josephson junction.

The assumptions for the specific boundary conditions considered above to simplify the solution are not crucial. Thus, the analysis can be extended to more general case. For instance, the analysis can be extended to the case when the loss term in (39) has a more general form. Also, it is of interest to find the conditions when nonlinear dissipation is obtained in the case of the reduced Maxwell-Bloch equations. Additionally, our developed approach can be applied, for example, to the flutter-like behavior of planar bodies falling in liquids where such nonlinear loss term in form of (39) are observed (Belmonte, Eisenberg and Moses).

Discussion

In this paper, we introduced an approach for finding differential equations that are known to be solvable (meaning integrable). We have showed how a generalized set of differential equations can be constructed based on simple generic equation with solution preserving condition. The reasoning for such condition is its practical applicability, which was shown in the following sections. Three examples were selected, Rayleigh-van der Pol equation, the reduced Maxwell Bloch and the reduced Sine Gordon equations, to show the applicability and efficiency of this approach. It may be stated that this approach is very effective in finding solutions for wide classes of problems encountered in real life. Obviously, practical applicability of solutions requires its stability. In our considered examples, the example in Section 2 ("Mathematical Model") seems to be stable by the definition, while the example considered in application 2 of Section 3 ("Results and Applications") seems to be stable as it is reported to be observed in the experiment. The stability of the solution considered in the application 1 of the Section 3 requires further analysis.

It is interesting to notice that during the procedure of generalizing, we impose the condition of invariance of the solution (at least for one variable) of the system under transformations of the coordinates, in particular, the global transformations of the coordinates (26). The requirements to satisfy these imposed conditions leads to the necessity for introduction of additional terms, which can be interpreted in our application examples as damping forces, in generally non-linear. Such procedure looks like the known approach used in Gauge field theory (Bailin and Love), in which the dynamics of the system is invariant under transformations of coordinates (Gauge transformation). The Gauge transformations require the introduction of new terms, called the compensation terms.

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