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Analyzing Oscillations on a Bridge

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ABSTRACT

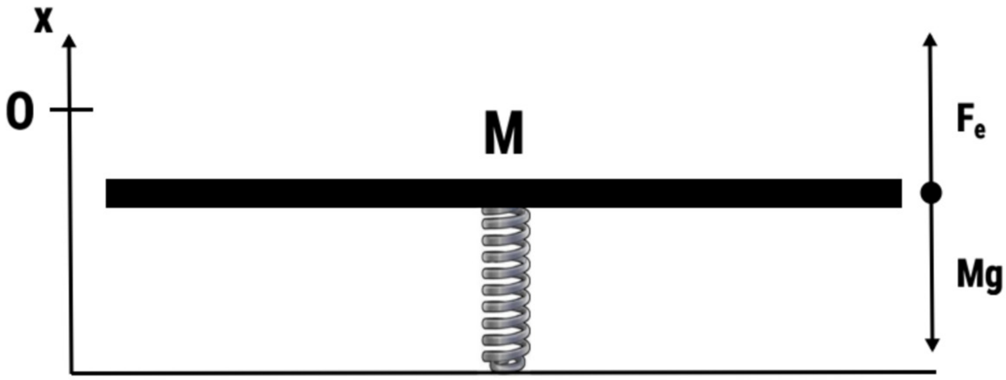
When people cross bridges, they create vibrations on these bridges because of the impulses they exert. In some cases, vertical oscillations have caused bridges to collapse due to a phenomenon called resonance. This paper utilizes simple harmonic motion to analyze the oscillations of bridges to create a mathematical model determining when certain bridges collapse. The paper first utilizes Hooke's Law and Newton's Second Law to create a second order differential equation of the motion of the bridge whose solution is a sine wave. Substituting in values for momentum for a singular impulse on the bridge, the paper then solves for the amplitude after the impulse. Adapting this model for impulses at different times and plotting the output graphs using Python, the bridge is shown to exhibit resonance and suggestions for damping are made.

Objective

The purpose of this research is to create a mathematical model of the vertical oscillations of a bridge when impulses are applied to it at different times. By understanding the number and frequency of impulses on the bridge, the model can be used to determine certain amplitudes where the bridge could potentially collapse. This research is important as civil engineers will be able to more accurately design damping systems to prevent the bridge from collapsing.

Introduction

Bridges have been built essentially since the start of humanity. From cavemen stacking logs across rivers to Romans building arch bridges, civil engineers have always designed new bridges to transport people and goods across bodies of water, roads, and valleys (Mishra, 2017). Bridges like the Golden Gate Bridge are even a part of popular culture; the bridge has been destroyed in movies like X-Men: the Last Stand, Godzilla, and Terminator: Genisys to demonstrate the power of a villain (Barker, 2021). There are a few different kinds of bridges including a beam bridge, truss bridge, cantilever bridge, arch bridge, suspension bridge, and cable-stayed bridge (Billington, 1999). In this paper, the theoretical bridge that is modeled is a beam bridge and is represented by a platform with a spring attached to the bottom of it as shown in Figure 1. The reason the spring is not a beam in this model is because a spring exhibits simple harmonic motion and can simplify the mathematical model for the oscillations of the platform.



The physics behind the collapse of bridges is often due to an effect called resonance. Resonance is where an object increases its amplitude when the frequency of the force applied is equal to the natural frequency of the object. The natural frequency is the frequency of an object in the absence of a driving force (Lim, 2018). For small oscillations, the platform in Figure 1 is a simple harmonic oscillator.

To describe the motion of an oscillator, we can instead consider a pendulum with a weight on the end of it, similar to a child on a swing. The pendulum has a natural frequency dependent on the pendulum's mass and length (Russell). Normally, the amplitude of the oscillations would decrease with time due to damping, which is why we have to keep pushing the child in the swing or else they will stop. If we assume that the bridge has no damping, the oscillations will grow forever due to resonance (Lim, 2018). A person pushing the child in the swing is an example of a driving force. The force is periodic, and it has the same frequency as the natural frequency of the pendulum of the swing-child system. This is an example of resonance.

The physics of the oscillation of a child is the same as the physics of the oscillation of a bridge. Similar to the pushes of a swing, the steps of marching soldiers on a bridge may act as periodic forces that cause oscillations with increasingly larger amplitudes that may cause a bridge to collapse without proper damping.

The mathematical model of a bridge without impulses

M is the mass of the platform. We ignore damping in this case. The forces acting on the platform are its weight, Mg, and the elastic spring force, F_e, which follows Hooke's Law. Assume the elastic spring force and the gravitational force is 0. This is where x is 0; the height of the platform is the equilibrium height. When the platform is not at the equilibrium height, the elastic spring force points in the opposite direction of the displacement. This is to say if the platform is below the equilibrium height, the elastic spring force points up, and if the platform is above the equilibrium height, the force points down.

The elastic spring force is defined as $F_e = kx$, where k is the spring constant, and k is always positive. Now, assume that the gravitational force is acting on the bridge. The net force on the bridge, defined as the sum of all forces acting on the bridge, is comprised of the elastic spring force and the gravitational force where $F_{net} = -kx - Mg$. Newton's Second Law shows

$$-kx - Ma = Mg, (1)$$

where t denotes time, and x is the displacement of the platform. Defining v and a as the velocity and acceleration of the platform, given by the first and second derivatives of x respectively, we can rewrite equation 1 as the following:

$$-kx - Mx'' = Mg. (2)$$

Equation 2 is a differential equation where the unknown is $x(t)$. When the platform is in static equilibrium (it is not moving), the acceleration (x'') is zero. For this reason, if we substitute 0 for x'' in equation 2, we get $-kx - Mg = 0$ and $Mg = -kx$. The constant function $x(t) = -Mg/k$ is a solution where the platform is not moving, and the elastic force is equal to the weight in opposite directions. We can now define a new frame of reference $z(t)$ where the platform rests at equilibrium with the force of gravity applied. Because z is yet another position function, $z' = x'$ and $z'' = x''$. $z(t)$ then becomes

$$z(t) = x(t) + \frac{Mg}{k}. \ (3)$$

z is 0 where x is $-Mg/k$ because $-\frac{Mg}{k} + \frac{Mg}{k} = 0$. With algebraic manipulation and defining $\omega = \sqrt{k/M}$, equation 2 becomes

$$-k\left(z - \frac{Mg}{k}\right) - Mg = Mz''$$
$$-kz + Mg = Mz''$$
$$-\frac{kz}{M} = z''$$
$$z'' + \frac{kz}{M} = 0$$
$$z'' + \omega^2 z = 0 \ (4)$$

There are an infinite number of solutions to equation 4. The solutions are

$$z = A\sin(\omega t + \phi) \ (5)$$

where A and ϕ are constants, determined by the initial position and velocity of the system where A is the amplitude of the platform.

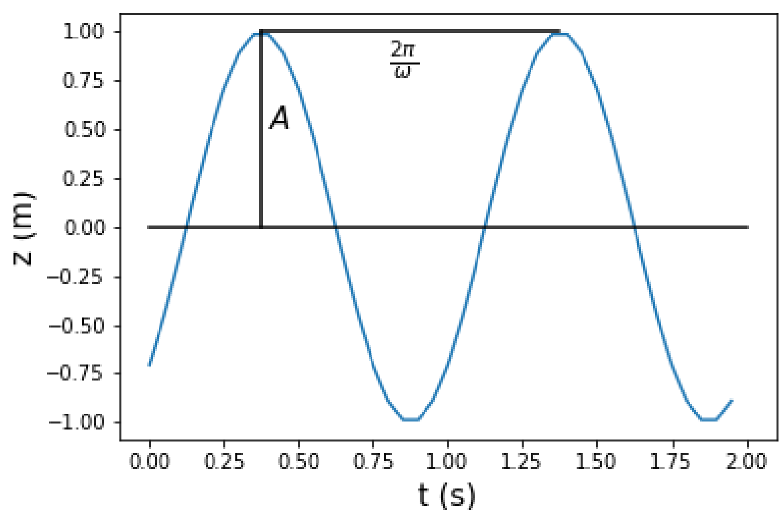


Figure 2 shows a plot of the solution when $A = 1$, $\omega = 2\pi$, and $\phi = -\frac{\pi}{4}$. One thing to note is that the platform will oscillate forever in this model because damping is neglected. The frequency $\omega = \sqrt{k/M}$ is the natural frequency of the platform, and the amplitude only depends on the initial conditions.

Effect of an impulse on the dynamics of the platform

An impulse is defined as a certain amount of mass in motion or a change in force applied over time (Khan Academy). The impulse in this case would be equivalent to a soldier stepping on a bridge exerting a constant force on it for a short period of time. We can model a soldier's footsteps as an impulse, $-J$, defined as

$$m_2 v_2 - m_1 v_1 = -J. \ (6)$$

Equation 6 simply demonstrates that the change in momentum of an object is equal to its impulse. If the platform is subjected to a single impulse at time t_1, equation 5 is now only correct for t where t does not equal t_1. Because the model does not account for an impulse, when the force is applied at t_1, the equation is not correct.

Therefore, we can rewrite equation 5 using the constants 1 and 2 for before and after the impulse on the platform, yielding two separate equations:

$$z(t) = \begin{cases} A_1 \sin(\omega t + \phi_1) \;\; if\; t < t_1 \\ A_2 \sin(\omega t + \phi_2) \;\; if\; t > t_1 \end{cases} \quad (7)$$

We have two boundary conditions of this system. z must be continuous, and the velocities of $z'(t)$ must satisfy equation 6. Because z is continuous and the change in momentum is $-J$ at time t_1, the constants A_2 and ϕ_2 can be found in terms of A_1 and ϕ_1. We can first set up this problem knowing that the position z before and after the impulse is the same.

$$A_1 \sin(\omega t_1 + \phi_1) = A_2 \sin(\omega t_1 + \phi_2)$$

We can find the velocity of $z(t)$ by taking the derivative of equation 7.

$$v_2 = \frac{d}{dt} A_2 \sin(\omega t_1 + \phi_2) = \omega A_2 \cos(\omega t_1 + \phi_2)$$
$$v_1 = \frac{d}{dt} A_1 \sin(\omega t_1 + \phi_1) = \omega A_1 \cos(\omega t_1 + \phi_1)$$

Because the mass of the platform is the same before and after the impulse, the masses are all the same: $m_1 = m_2 = M$. If we substitute these expressions into equation 6, we get

$$M\omega A_2 \cos(\omega t_1 + \phi_2) - M\omega A_1 \cos(\omega t_1 + \phi_1) = -J$$
$$M\omega A_2 \cos(\omega t_1 + \phi_2) = M\omega A_1 \cos(\omega t_1 + \phi_1) - J \quad (8)$$

Dividing out the $M\omega$ and defining $I = J/M\omega$, our new equation is

$$A_2 \cos(\omega t_1 + \phi_2) = A_1 \cos(\omega t_1 + \phi_1) - I.$$

Our two boundary conditions for equation 7 are then given by

$$A_2 \sin(\omega t_1 + \phi_2) = A_1 \sin(\omega t_1 + \phi_1) \quad (9)$$

and

$$A_2 \cos(\omega t_1 + \phi_2) = A_1 \cos(\omega t_1 + \phi_1) - I. \quad (10)$$

We can square both equations and add them to get the following:

$$A_2^2\left[\sin^2(\omega t_1 + \phi_2) + \cos^2(\omega t_1 + \phi_2)\right] = A_1^2\left[\sin^2(\omega t_1 + \phi_1) + \cos^2(\omega t_1 + \phi_1)\right] - 2A_1 I\cos(\omega t_1 + \phi_1) + I^2. \quad (11)$$

Given the trigonometric identity $\sin^2 x + \cos^2 x = 1$, equation 11 can be rewritten as the following:

$$A_2^2 = A_1^2 - 2A_1 I \cos(\omega t_1 + \phi_1) + I^2. \quad (12)$$

Taking the square root of equation 12, we arrive at the following expression for A_2:

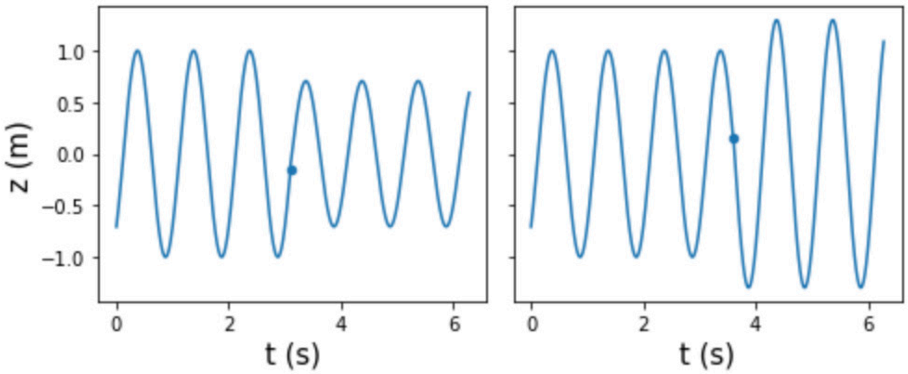
$$A_2 = \sqrt{A_1^2 - 2A_1 I \cos(\omega t_1 + \phi_1) + I^2}. \quad (13)$$

Our next goal is to find ϕ_2 now that we have A, starting with equation 9.

$$A_2 \sin(\omega t_1 + \phi_2) = A_1 \sin(\omega t_1 + \phi_1)$$
$$\sin(\omega t_1 + \phi_2) = A_1 \sin(\omega t_1 + \phi_1)/A_2$$
$$\sin^{-1}(\sin(\omega t_1 + \phi_2)) = \sin^{-1}(A_1 \sin(\omega t_1 + \phi_1)/A_2)$$
$$\omega t_1 + \phi_2 = \sin^{-1}(A_1 \sin(\omega t_1 + \phi_1)/A_2)$$

Subtracting ωt_1 from both sides, we get the following expression for ϕ_2:

$$\phi_2 = \begin{cases} \sin^{-1}(A_1 \sin(\omega t_1 + \phi_1)/A_2) - \omega t_1 \;\; if\, A_1 \cos(\omega t_1 + \phi_1) - I > 0 \\ \pi - \sin^{-1}(A_1 \sin(\omega t_1 + \phi_1)/A_2) - \omega t_1 \;\; if\, A_1 \cos(\omega t_1 + \phi_1) - I < 0\,. \end{cases} \quad (14)$$



Two examples are depicted in Figure 3 plotting the graph of $z(t)$. The dot shows when the impulse is applied. The left figure has the parameters $A_1 = 1$, $\omega = 2\pi$, $\phi_1 = -\pi/4$, $I = 0.3$, and $t_1 = 3.1$. The right figure has the parameters $A_1 = 1$, $\omega = 2\pi$, $\phi_1 = -\pi/4$, $I = 0.3$, and $t_1 = 3.6$. In Figure 3a, because the platform was moving up when receiving the downward impulse, the amplitude decreased in the left figure. In Figure 3b, because the platform was moving down when receiving the downward impulse, the amplitude increased in the left figure. The impulse increased the velocity of the platform.

The mathematical model of a platform with multiple impulses

We can expand our current mathematical model to account for the effect of multiple simultaneous impulses on the platform. Using the soldier analogy, the soldiers would walk in unison at a constant velocity on the bridge, exerting an impulse $-J$ each time they land a foot on it. Let T be a positive number where $t_n = nT$. At each time t_n, the platform is subjected to an impulse for each non-negative integer n. For example, the time interval T between each impulse could be 0.2 seconds, so the platform would be subjected to an impulse at 0 seconds, 0.2 seconds, 0.4 seconds, 0.6 seconds, etc. Equation 5 can be written as the following:

$$z = A_n \sin(\omega t + \phi_n), \text{ where } t_{n-1} < t < t_n. \quad (15)$$

Similarly, we can generalize equations 13 and 14 to get the following:

$$A_{n+1} = \sqrt{A_n^2 - 2A_n I \cos(\omega t_n + \phi_n) + I^2}$$

and

$$\phi_{n+1} = \sin^{-1}(\mathrm{A_n}\sin(\omega t_n + \phi_n)/A_{n+1}) - \omega t_n$$

Our starting condition is $A_0 = 0$ because the platform is not moving before the first impulse. We also have $\phi_{n+1} = 0$ and $A_{n+1} = 0$, otherwise

$$\phi_{n+1} = \pi - \sin^{-1}(\mathrm{A_n}\sin(\omega t_n + \phi_n)/A_{n+1}) - \omega t_n \ \ if\ A_n \cos(\omega t_n + \phi_n) - I < 0$$

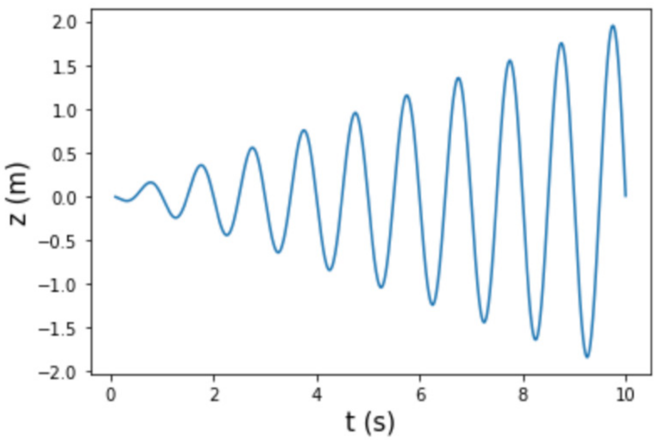


Figure 4 shows a plot of $z(t)$. In this example, $\omega = 2\pi$, $I = 0.2$, and $T = 1$. Because the function's amplitude grows linearly with time, $z(t)$ is not a periodic function of t. The figure corresponds to a resonant case. The bridge would then collapse at a certain amplitude because of the many impulses.

Discussion

There are a few key findings throughout the paper. The paper first utilizes Hooke's Law and Newton's Second Law to create a second order differential equation of the motion of the bridge whose solution is a sine wave. Substituting in values for momentum for a singular impulse on the bridge, the paper then solves for the amplitude and ϕ after the impulse. Adapting this model for impulses at different times and plotting the output graphs, the bridge is shown to exhibit resonance. A potential extension of this research would be to include a model for damping similar to equation 2 where $Mx" + cx' + kx = -Mg$ where c is a damping coefficient that can be found to intentionally help design bridges to prevent resonance (García et al., 2003). The limitations of these models are clear and illuminate opportunities for more experimentation in the future. Primarily, the resonant models only account for one impulse applied at a time. This, frankly, is quite unrealistic in the real world given many people walk across bridges and exert impulses on the bridge of different sizes at different times. A model that could account for the oscillations while allowing for various sized impulses at different times could more accurately model soldiers walking across a bridge. Furthermore, the model is not fully realistic for most types of bridges. Although the beam bridge is mentioned in the introduction, given this bridge is supported on two sides, the bridge is not free to move up and down, rather only in the center, creating a U-shape. We do not analyze how the bridge would bend or to what extent a bridge could bend in the center before snapping (Gou et al., 2018). Also, we leave out critical aspects of bridge-building including the bridge moving side to side simultaneously, damped oscillators, torsion, and aeroelasticity. Primarily, most bridges have dampening mechanisms to prevent them from oscillating to the point of collapse, particularly during a resonant case described above. We also neglect air resistance and its effect on the torsion (twisting) of the bridge. Aeroelastic "flutter" is when a structure has positive feedback between the body's deflection (in this case air) and the force exerted by the fluid flow (air as well) (Deshpande, 2016). A famous example of "flutter" caused the Tacoma Narrows bridge to fall among many other examples (Arioli & Gazzola, 2014).

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