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Differential Analysis on A Simple Pendulum System

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ABSTRACT

Given a system of a simple pendulum, we analyze the solution that is obtained from the nonlinear second-order differential equation that characterizes the pendulum and compare it to the corresponding small-signal linear approximation of the nonlinear system. In addition, the properties that are present for the simple pendulum system like the exact period of the pendulum system is derived and compared to that of the small-signal linearization approximate system. The time per unit angular displacement for the pendulum system is proposed, derived, and discussed. During this process, an approach is presented to compute the incomplete elliptic integral of the first kind. Finally, the concept of the daily average periodicity of the pendulum is proposed and several considerations are provided in regulating the daily average periodicity.

Introduction

During my sophomore year of high school, in a physics class, I learned how pendulums operated and how one could describe its motions with a set of precise equations. I became very interested in how from a calculus standpoint, one could address the assumptions that are commonly made for pendulums such as the mass being able to reach a constant maximum height indefinitely. And therefore, when I determined the second order nonlinear differential equation, and realized that it takes care of such assumptions, my curiosity only increased.

While using a free body diagram on the mass and using Newton's 2nd and 3rd laws to build balanced force equations, a second-order differential equation arises. While this is not a linear differential equation, with some approximation, the nonlinear differential equation becomes a second-order linear differential equation, whose solution confirms that the pendulum's movement is indeed approximately sinusoidal and periodic.

However, in this research paper, I wish to not simply stop at this differential equation as it is standard in most textbooks, but more importantly, to answer more profound questions and explore special circumstances that arise from such an equation. Instead of focusing on the differential equation itself, I will focus on the consequences that arise. A motivation of some questions is listed below:

Suppose the pendulum is moved away from its vertical, static position for some angle and held stationary and then released. For a period of observation time, how much time difference is there between the actual time the pendulum takes to reach the maximum swing positions, vertical position and the corresponding times from the expected solutions resulting from an approximation?

Watching for another similar period of observation time, does the time difference increase or decrease? And if so, by how much? Otherwise, does it remain unchanged?
How does the increment between two consecutive observations vary? Do patterns exist in these observations?

Can the above quantities be computed or estimated to a desired precision mathematically?

Since the pendulum does not exhibit perfect sinusoidal periodicity even though it does exhibit periodicity, is it possible to strike the pendulum in specific time intervals to make it move more perfectly sinusoidal periodically? What is the amount of force that should be applied and how often the force should be applied so that the original periodicity would be maintained?

These are the set of questions I would like to explore and seek various conclusions by branching into different directions from the fundamental pendulum equation; that is, the 2nd order differential equation and its small-signal linearization.

Simple Pendulum System

A simple gravity pendulum system is shown in Fig. 1. and its corresponding force analysis of the free-body diagram is drawn in Fig. 2. The applications of Newton's Second and Third Laws are used in which a second order nonlinear differential equation is resulted [1,2,3]:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} sin\theta = 0 \qquad \ldots\ldots (1)$$

where g is a constant representing acceleration due to gravity, l is a constant representing the length of the pendulum, and θ is the angular displacement (which changes with time t, hence a function of time t); that is, we have the initial position θ at time $t = t_0 = 0$: $\theta(t_0) = \theta_0$.

The objectives of this research are to

- Study the dynamic behavior of $\theta(t)$.
- Study the difference between $\theta(t)$ from the above equation and $\theta_d(t)$ from the desired, ideal pendulum system over time, as specified below in Eq. (2).
- Study what can be experimentally done to make the dynamic behavior of $\theta(t)$ to be as perfect as that of Eq. (2).

A desired ideal pendulum system, which is represented by Eq. (1) with $sin\theta$ approximated as θ using a small angle approximation, is given by a second-order linear differential equation:

$$\frac{d^2\theta_d}{dt^2} + \frac{g}{l}\theta_d = 0 \qquad \ldots\ldots (2)$$

where θ_d is the angular displacement (which changes with time t, hence a function of time t).

Note that while Eq. (2) is a linear second-order differential equation, whose analytical solution is readily available, Eq. (1) is nonlinear second-order differential equation, whose analytical solution is not available [4].

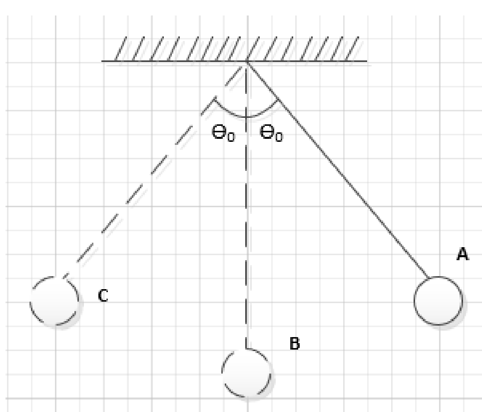

Fig. 1. Simple Pendulum System

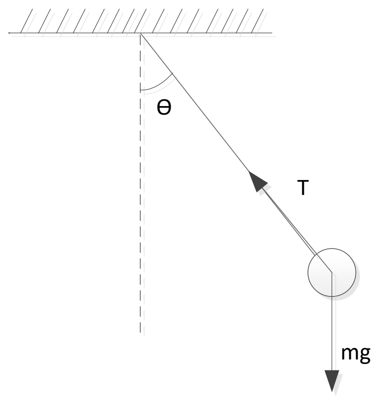

Fig. 2. Free Body Diagram

In Fig. 2, the mathematical representation of the dynamics of the pendulum is given by the following using Newton's laws:

$$\begin{cases} mgsin\theta = ma \\ a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -\frac{d^2(l\theta)}{dt^2} \end{cases} \quad (3)$$

where a is the tangential acceleration of the pendulum, s is the displacement of the tangential movement, and v is the tangential velocity.

Considering the length of the pendulum being a constant, independent of θ, then we have $\frac{d^2(l\theta)}{dt^2} = l\frac{d^2\theta}{dt^2}$. With some algebraic manipulation, the following equation is obtained:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}sin\theta = 0 \quad (4)$$

If l is a function of θ, which can be represented as $l = l_0 + k(\theta)$ where $k(\theta)$ is a function of θ with $k(0) = 0$, then Eq. (3) becomes

$$l\frac{d^2\theta}{dt^2} + \theta\frac{d^2l}{dt^2} + 2\frac{dl}{dt}\frac{d\theta}{dt} + gsin\theta = 0 \quad (5)$$

Note that mathematically, for all length functions $l(\theta)$ of degree N, there exist no such $l(\theta)$ such that Eqs. (5) can become a linear 2nd order differential. While the goal would be to vanish the $gsin\theta$ term, this is indeed impossible for any length function $l(\theta)$.

The rest of this paper is organized as follows: In Section III, the exact periodicity of the pendulum is computed, and the properties of the pendulum are investigated. The approximation of the pendulum's angular displacement is discussed using Fourier series expansion and the corresponding approximate time periodicity is computed in Section IV. A comparison is made between the exact periodicity and its angular displacement of the pendulum and their corresponding quantities resulting from the typical approximation for the perfect sinusoidal periodicity. In addition, based on the principles of physics, an innovative way to approximate the computation of the incomplete elliptic integral of the first kind is presented. In Section V, I discuss the possibility of a periodical strike on the pendulum, in which a daily average periodicity that is identical to the periodicity of Eq. (2) can be achieved. Finally, the concluding remarks are drawn in Section VI.

PROPERTIES OF THE PENDULUM

The exact periodicity of the pendulum and the angular displacement are computed in this section. For ease of description of the movement of the pendulum, denote the farthest right position of the pendulum, as shown in Fig. 1, as point A, the farthest left position of the pendulum, as point C, and the vertical position as point B. Therefore, a full cycle of the movement of the pendulum can be described as $A \rightarrow B \rightarrow C \rightarrow B \rightarrow A$. $T_{A\rightarrow B}$ denotes the time taken over the trajectory $A \to B$, and similarly, $T_{B\to C}$ the time taken over $B \to C$, and $T_{C\to B}$ is the time taken over $C \to B$, and $T_{B\to A}$ is the time taken over $B \to A$.

Note that $v = \frac{ds}{dt} = l\frac{d\theta}{dt}$ from Eq. (3). Then, let point P be on the trajectory $A \to B$, with the corresponding angle θ.

Since energy is conserved from A to P, the energy conservation leads to the following equation:
$\frac{1}{2}mv^2 = mg(lcos\theta - lcos\theta_0)$, which reduces to
$v = -\sqrt{2gl(cos\theta - cos\theta_0)}$.
Note that the sign is negative because its motion is leftward from A to P.
Next we have,

$$\frac{d\theta}{dt} = \frac{v}{l} = -\sqrt{\frac{2g}{l}(cos\theta - cos\theta_0)}. \qquad (6)$$

Accordingly,

$$\frac{dt}{d\theta} = -\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}. \qquad (7)$$

Thus, the time taken over the trajectory $A \to D$ can be represented by

$$t = \int_{\theta_0}^{\theta} -\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}\,d\theta = \int_{\theta}^{\theta_0}\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}\,d\theta \qquad (8)$$

Then $T_{A\to B} = \int_0^{\theta_0}\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}\,d\theta$.

Similarly for the other expressions, we have

$$T_{B\to C} = \int_0^{-\theta_0} -\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}\,d\theta = \int_0^{\theta_0}\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}\,d\theta.$$

$$T_{C\to B} = \int_{-\theta_0}^{0}\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}\,d\theta = \int_0^{\theta_0}\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}\,d\theta.$$

$$T_{B\to A} = \int_0^{\theta_0}\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}\,d\theta.$$

From the previous equations, it is important to observe that

$$T_{A\to B} = T_{B\to C} = T_{C\to B} = T_{B\to A} = \int_0^{\theta_0}\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}\,d\theta. \qquad (9)$$

The time $T_{B\to A}$ over the trajectory of the pendulum from point A to point B, to point C, then back to point B, and then back to point A is determined as

$$T_{A\to B\to C\to B\to A} = T_{A\to B} + T_{B\to C} + T_{C\to B} + T_{B\to A} = 4\int_0^{\theta_0}\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}\,d\theta. \qquad (10)$$

Let $T = T_{A\to B\to C\to B\to A}$. Then from Eqs. (9) and (10), we obtain

$$T_{A\to B} = T_{B\to C} = T_{C\to B} = T_{B\to A} = \frac{T}{4}. \qquad (11)$$

Once the pendulum returns to point A, it then repeats the course of movement $A \to B \to C \to B \to A$. Therefore, the pendulum's movement is periodic with the period being $T = T_{A\to B\to C\to B\to A}$. Mathematically, we have

$$\theta(t+T) = \theta(t) \text{ for any } t \geq 0. \qquad (12)$$

Note that the angular displacement is nonnegative on the trajectories $A \to B$ and $B \to A$, and it is nonpositive on the trajectories $B \to C$ and $C \to B$.

In the following, for a given positive angle θ_c ($\theta_c > 0$), the corresponding times are determined for the angular displacement $\theta_c$on the trajectories $A \to B$ and $B \to A$, and $-\theta_c$ on the trajectories $B \to C$ and $C \to B$. In addition, we would like to explore how these times are related among themselves.

Again, from Eq (7), t can be expressed in terms of θ_c by integrating both sides of the equation:

$$t_1 = \int_{\theta_0}^{\theta_c} -\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}\,d\theta = \int_{\theta_c}^{\theta_0}\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}\,d\theta, \qquad (13)$$

where $(t_1, \theta(t_1)) = (t_1, \theta_c)$ represents the time point t_1 and the corresponding angular displacement $\theta(t_1) = \theta_c$ of the pendulum over the trajectory $A \to B$.

Similarly, $(t_2, \theta(t_2)) = (t_2, -\theta_c)$ represents the time point t_2 and the corresponding angular displacement $\theta(t_2) = -\theta_c$ of the pendulum over the trajectory $B \to C$. Similarly, $(t_3, \theta(t_3)) = (t_3, -\theta_c)$ represents the time point t_3 and the corresponding angular displacement $\theta(t_3) = -\theta_c$ of the pendulum over the trajectory $C \to B$. Similarly, $(t_4, \theta(t_4)) = (t_4, \theta_c)$ represents the time point t_4 and the corresponding angular displacement $\theta(t_4) = \theta_c$ of the pendulum over the trajectory $B \to A$.

t_2, t_3, and t_4 can be determined as follows:

$$t_2 = \int_{\theta_0}^{0} -\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}d\theta + \int_{0}^{-\theta_c} -\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}d\theta = \frac{T}{4} + \int_{0}^{\theta_c} \sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}d\theta$$
$$= \frac{T}{4} + \int_{0}^{\theta_0} \sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}d\theta - \int_{\theta_c}^{\theta_0} \sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}d\theta = \frac{T}{4} + \frac{T}{4} - t_1 = \frac{T}{2} - t_1 \quad (14)$$

$$t_3 = \int_{\theta_0}^{-\theta_0} -\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}d\theta + \int_{-\theta_0}^{-\theta_c} \sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}d\theta = \frac{T}{2} + \int_{\theta_0}^{\theta_c} -\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}d\theta$$
$$= \frac{T}{2} + \int_{\theta_c}^{\theta_0} \sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}d\theta = \frac{T}{2} + t_1 \quad (15)$$

$$t_4 = \int_{\theta_0}^{-\theta_0} -\sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}d\theta + \int_{-\theta_0}^{0} \sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}d\theta + \int_{0}^{\theta_c} \sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}d\theta = \frac{T}{2} + \frac{T}{4} + \int_{0}^{\theta_c} \sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}d\theta$$
$$= \frac{3T}{4} + \int_{0}^{\theta_0} \sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}d\theta - \int_{\theta_c}^{\theta_0} \sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}d\theta$$

$$= \frac{3T}{4} + \frac{T}{4} - \int_{\theta_c}^{\theta_0} \sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}\, d\theta$$
$$= T - t_1 \qquad (16)$$

From Eqs. (12) – (16), the following conclusions can be drawn, which are regarded as the properties of the pendulum system:

- $\theta\left(\frac{T}{2} - t\right) = -\theta(t)$ for $0 \leq t \leq \frac{T}{2}$.
- $\theta\left(\frac{T}{2} + t\right) = -\theta(t)$ for $0 \leq t \leq \frac{T}{2}$.
- $\theta(T - t) = \theta(t)$ for $0 \leq t \leq T$.
- $\theta(T + t) = \theta(t)$ for $t \geq 0$.

It is noteworthy that a cosine function like $\theta(t) = \cos\left(\frac{2\pi}{T} t\right)$ that characterizes harmonic motion also possesses the above listed properties.

THE EXACT PERIODICITY, TIMES PER ANGULAR DISPLACEMENT, AND THEIR CORRESPONDING APPROXIMATES

From Eq. (10), the pendulum's period is given by $T = 4\int_0^{\theta_0} \sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}\, d\theta = 4\sqrt{\frac{l}{2g}} \int_0^{\theta_0} \sqrt{\frac{1}{cos\theta - cos\theta_0}}\, d\theta$.

Let $\alpha = \arcsin\left(\frac{sin\frac{\theta}{2}}{sin\frac{\theta_0}{2}}\right)$. Then for $0 \leq t \leq \theta_0$, $0 \leq \alpha \leq \frac{\pi}{2}$.

Then, the above equation is converted to

$$T = 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \sqrt{\frac{1}{1 - \sin^2\frac{\theta_0}{2}\sin^2\alpha}}\, d\alpha. \qquad (17)$$

Note that in Eq. (17), the integral is of the form of an elliptic integral [5]. With the help of the Legendre polynomial solution for the elliptic integral, Eq. (17) can be rewritten as

$$T = 2\pi\sqrt{\frac{l}{g}} \sum_{k=0}^{\infty} \left(\frac{(2k)!}{(2^k k!)^2}\right)^2 \sin^{2k}\frac{\theta_0}{2}. \qquad (18)$$

With $\sin\frac{\theta_0}{2}$ expanded into a power series, Eq. (18) becomes

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \cdots\right) \qquad (19)$$

Note that the period of the system given by Eq. (2), that is approximately obtained from Eq. (1) when the magnitude of the swing of the pendulum is small, is given by

$$T_d = 2\pi\sqrt{\frac{l}{g}}. \qquad (20)$$

Comparing Eqs. (18) and (20), we obtain

$$T - T_d = 2\pi\sqrt{\frac{l}{g}} \sum_{k=1}^{\infty} \left(\frac{(2k)!}{(2^k k!)^2}\right)^2 \sin^{2k}\frac{\theta_0}{2} > 0 \text{ for } \theta_0 > 0.$$

Thus, $T > T_d$ for $\theta_0 > 0$.

This implies that the typical approximate for the period of the pendulum is smaller than its actual period. In the remainder of this section, we focus on the variance in the time it takes for the pendulum to travel its corresponding per unit angular displacement. Due to the properties as listed in the end of last section, it is necessary to continue this analysis for the first quarter cycle of the pendulum.

Let the angular displacement θ_0 be evenly divided into N pieces. Let the according times traveling each $\frac{\theta_0}{N}$ angular displacement be $t_1^N, t_2^N, \ldots, t_N^N$. Then t_k^N where $1 \leq k \leq N$ is given by

$$t_k^N = \int_{\frac{(N-k)\theta_0}{N}}^{\frac{(N-k+1)\theta_0}{N}} \sqrt{\frac{l}{2g(cos\theta - cos\theta_0)}}\, d\theta.$$

Let $\alpha = \arcsin\left(\frac{sin\frac{\theta}{2}}{sin\frac{\theta_0}{2}}\right)$.

Then for $\frac{(N-k)\theta_0}{N} \le \theta \le \frac{(N-k+1)\theta_0}{N}$, $\arcsin\left(\frac{sin\frac{(N-k)\theta_0}{2N}}{sin\frac{\theta_0}{2}}\right) \le \alpha \le \arcsin\left(\frac{sin\frac{(N-k+1)\theta_0}{2N}}{sin\frac{\theta_0}{2}}\right)$.

Let $\alpha_{N-k} = \arcsin\left(\frac{sin\frac{(N-k)\theta_0}{2N}}{sin\frac{\theta_0}{2}}\right)$. Then the above equation is converted to

$$t_k^N = \sqrt{\frac{l}{g}} \int_{\alpha_{N-k}}^{\alpha_{N-k+1}} \sqrt{\frac{1}{1-\sin^2\frac{\theta_0}{2}\sin^2\alpha}}\, d\alpha. \qquad (21)$$

Similarly for the system characterized by Eq. (2), the angular displacement is given by

$$\theta(t) = \theta_0 \cos\left(\frac{2\pi}{T_d}t\right). \qquad (22)$$

For $0 \le t \le \frac{T_d}{4}$, $t = \frac{T_d}{2\pi}\arccos\left(\frac{\theta(t)}{\theta_0}\right)$.

For the harmonic motion characterized by Eq. (18), let the times for traveling each $\frac{\theta_0}{N}$ angular displacement be denoted as $t_{d,1}^N, t_{d,2}^N, \ldots, t_{d,N}^N$. Then $t_{d,k}^N$ can be determined by

$$t_{d,k}^N = \frac{T_d}{2\pi}\left(\arccos\left(\frac{N-k}{N}\right) - \arccos\left(\frac{N-k+1}{N}\right)\right), \quad (23)$$

where $1 \le k \le N$.

Note that Eq. (21) can be rewritten as

$$t_k^N = 4\sqrt{\frac{l}{g}}\left(\int_0^{\alpha_{N-k+1}} \sqrt{\frac{1}{1-\sin^2\frac{\theta_0}{2}\sin^2\alpha}}\, d\alpha - \int_0^{\alpha_{N-k}} \sqrt{\frac{1}{1-\sin^2\frac{\theta_0}{2}\sin^2\alpha}}\, d\alpha\right).$$

Note that when θ_0 is sufficiently small, then t_k^N and $t_{d,k}^N$ are approximately identical, that is, $t_k^N = t_{d,k}^N$. That is,

$$\sqrt{\frac{l}{g}} \int_{\alpha_{N-k}}^{\alpha_{N-k+1}} \sqrt{\frac{1}{1-\sin^2\frac{\theta_0}{2}\sin^2\alpha}}\, d\alpha = \frac{T_d}{2\pi}\left(\arccos\left(\frac{N-k}{N}\right) - \arccos\left(\frac{N-k+1}{N}\right)\right). \qquad (24)$$

It is noteworthy that since $\arccos\left(\frac{N-k}{N}\right)$ is known for $1 \le k \le N$, and $\alpha_0 = 0$, then $\sqrt{\frac{l}{g}} \int_0^{\alpha_{N-k+1}} \sqrt{\frac{1}{1-\sin^2\frac{\theta_0}{2}\sin^2\alpha}}\, d\alpha = \sqrt{\frac{l}{g}} \int_0^{\alpha_{N-k}} \sqrt{\frac{1}{1-\sin^2\frac{\theta_0}{2}\sin^2\alpha}}\, d\alpha + \frac{T_d}{2\pi}\left(\arccos\left(\frac{N-k}{N}\right) - \arccos\left(\frac{N-k+1}{N}\right)\right)$ can be iteratively determined, first for $\sqrt{\frac{l}{g}} \int_0^{\alpha_1} \sqrt{\frac{1}{1-\sin^2\frac{\theta_0}{2}\sin^2\alpha}}\, d\alpha$ (where $k = N$), then $\sqrt{\frac{l}{g}} \int_0^{\alpha_2} \sqrt{\frac{1}{1-\sin^2\frac{\theta_0}{2}\sin^2\alpha}}\, d\alpha$ (where $k = N-1$), and eventually $\sqrt{\frac{l}{g}} \int_0^{\alpha_N} \sqrt{\frac{1}{1-\sin^2\frac{\theta_0}{2}\sin^2\alpha}}\, d\alpha$ (where $k = 1$). In other words, the above approach provides a way to approximately compute the incomplete elliptic integral of the first kind.

AVERAGE DAILY PERIODICITY AND AGULARDISPLACEMENT WITH EXTERNAL MOMENTUM

While the periodicity of the simple pendulum system depends on the maximum angular displacement from the vertical line, the length of the pendulum, and the acceleration due to gravity, the periodicity as determined for the linear system resulting from small-signal linearization of the simple pendulum system does not depend on the maximum angular displacement. For ease of description, let us assume the clock associated with the former (actual pendulum) be denoted clock 1 and the clock associated with the latter (linearized pendulum) be denoted clock 2.

From the discussion in the previous section, clock 1 ticks slower than clock 2. Since we are trying to evaluate the difference of these two clocks on a 24-hour basis, it is natural to introduce the concept of average daily periodicity – which we define as follows:

$T_{avg} = \frac{n_1T_1+n_2T_2+\cdots+n_MT_M}{n_1+n_2+\cdots+n_M}$, where T_k denotes the exact period of the simple pendulum system with the corresponding maximum angular displacement θ_k for clock 1 and n_k designates the number of cycles associated with the period T_k; and n_k's and T_k's are such that $n_1T_1 + n_2T_2 + \cdots + n_MT_M = T_d(n_1 + n_2 + \cdots + n_M)$ where T_d represents the period of clock 2.

Accordingly, the average daily angular displacement for clock 1 is defined as:
$\theta_{avg} = \frac{n_1\theta_1+n_2\theta_2+\cdots+n_M\theta_M}{n_1+n_2+\cdots+n_M}$.

Note that $T_d(n_1 + n_2 + \cdots + n_M)$ represents the time of a typical 24-hour day. In other words, $T_d(n_1 + n_2 + \cdots + n_M)$ is fixed. Let this fixed number be denoted as D. Then $T_d(n_1 + n_2 + \cdots + n_M) = D$.

To illustrate the concept of the average daily periodicity, assume that the simple pendulum starts with the maximum angular displacement $\theta_1 = \theta_0$. From Eq. (19), $T_1 = 2\pi\sqrt{\frac{l}{g}}(1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \cdots)$. Obviously, $T_1 > T_d = 2\pi\sqrt{\frac{l}{g}}$. At this rate, clock 1 would be slower than clock 2 at the end of a 24-hour observation period.

Thus, it needs to speed up, which implies that at least one different period (say T_2) needs to be used during the observation period. Then $\frac{n_1T_1+n_2T_2}{n_1+n_2} = T_d$.

Since T_d is known from $T_d = 2\pi\sqrt{\frac{l}{g}}$, and $n_1 + n_2$ is also known from $n_1 + n_2 = \frac{D}{T_d}$, selection of n_1 would imply determination of n_2, which in turn implies determination of T_2, as $T_2 = \frac{D}{\frac{D}{T_d}-n_1} - \frac{n_1T_1}{\frac{D}{T_d}-n_1}$.

Note that since n_1 and n_2 are supposed to be integers, and θ_{avg}, in this case, $\theta_{avg} = \frac{n_1\theta_1+n_2\theta_2}{n_1+n_2}$, is supposed to be as close to both as possible to avoid significant change in maximum angular displacement to minimize the implementation of the external energy, then n_1 and n_2 should be about the same. Thus, if $\frac{D}{2T_d} - \left\lfloor\frac{D}{2T_d}\right\rfloor < 0.5$, then $n_1 = n_2 = \left\lfloor\frac{D}{2T_d}\right\rfloor$; otherwise $n_1 = \left\lfloor\frac{D}{2T_d}\right\rfloor + 1$, then $n_2 = \left\lfloor\frac{D}{2T_d}\right\rfloor$ where $\left\lfloor\frac{D}{2T_d}\right\rfloor$ is a floor function that represents the largest integer that is less than or equal to $\frac{D}{2T_d}$.

To minimize the number of adjustments in maximum angular displacement, if one 24-hour period has θ_1 for n_1 cycles followed by θ_2 for n_2 cycles, then the next 24-hour period continues to start with θ_2 for n_2 cycles followed by θ_1 for n_1 cycles, and the following 24-hour period continues to start with θ_1 for n_1 cycles followed by θ_2 for n_2 cycles, and so on and so forth.

In order to alter the periodicity of the pendulum, there are a few ways that can achieve this:

- Change the length of the pendulum when the bob reaches its maximum angular displacement.
- Change the maximum angular displacement by designing an appropriate collision (which essentially changes the speed of the bob to some desired value). To switch from the maximum angular displacement θ_1 to a larger maximum angular displacement θ_2, an impulse may be applied when the pendulum bob reaches the maximum angular displacement θ_1. The amount of impulse required is given by $m\sqrt{2gl(cos\theta_1 - cos\theta_2)}$, which represents the momentum change. Similarly, to switch from the maximum angular displacement θ_1 to a smaller maximum angular displacement θ_2, an impulse may be applied in the opposite direction when the pendulum bob reaches the angular displacement 0 (i.e., the vertical line). The amount of impulse required is given by $m\sqrt{2gl(cos\theta_2 - cos\theta_1)}$, which represents the momentum change.
- Place the pendulum system in a upwards or downwards accelerating environment with acceleration being a, which essentially changes the gravitational acceleration to $g + a$ or $g - a$. In turn, this changes the period of the pendulum system. An example of an environment is an upward or downward accelerating elevator.

Conclusions

This research paper dives into the several questions on a simple pendulum system I have had from my Physics course experience and initial interests. In typical Physics textbooks, the period of a simple pendulum system is obtained by applying small-signal linearization. The resulting approximate period no longer depends on the maximum angular displacement. In reality, the dynamic system that characterizes the simple pendulum system is actually nonlinear. This nonlinear system indeed exhibits periodic motion, but it is not harmonic motion. This paper studies the non-harmonic periodic dynamic behaviors and reveals some interesting properties of the simple pendulum system. In addition, this paper proposes the concept of average time per unit angular displacement and makes the comparison between the nonlinear nature of the simple pendulum system and its linear nature of the resulting approximate dynamic system. Furthermore, this research paper determines a new way of computing the incomplete elliptic integral. In addition, this paper introduces the concepts of average daily period and average daily angular displacement, and with that provides means to compensate the nonlinearity of the simple pendulum system in an average sense. This achieves desired accuracy when compared to the linear approximate system resulting from the nonlinear simple pendulum system.

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